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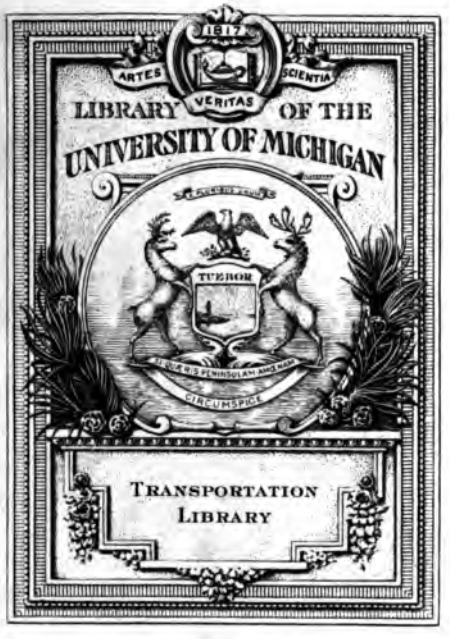
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THE THEORY OF DEFLECTIONS

AND OF

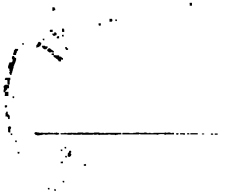
LATITUDES AND DEPARTURES,

WITH SPECIAL APPLICATION TO CURVILINEAR SURVEYS FOR ALIGNMENTS OF RAILWAY-TRACKS.

BY

ISAAC W. SMITH, C.E.,

Member of the American Society of Civil Engineers.



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PREFACE.

THE theory of latitudes and departures has been long in use in plane surveying, but the theory of deflections, although applied, to some extent, in railway surveys, has never, to my knowledge, been reduced to a system, or even considered in any mathematical treatise.

The object of this work is the application of these theories to the solution of problems in curvilinear surveys for alignments of railroad-tracks.

The greater portion of the problems were originated many years since, but were not published, because the demonstration of the principles on which the solutions of the problems are based is not possible under the ordinary definition of an angle, as a relation between two straight lines, in contradistinction to a deflection, considered as a relation between two directions.

To present the subject in an intelligible light, it has therefore been necessary to consider the theories of deflections, and of latitudes and departures, and this consideration has swelled the volume of the book much beyond my original intention.

Aside from this, the problems occupy a comparatively small space, and, with a few exceptions, are special applications of problems XVI. and XXVI., tables III. and XII., the one with relation to intersecting curves, and the other to curves connected by a common tangent of given length.

The same symbols for quantities, points, lines, and directions are used throughout, and the only departure from ordinary custom is in the symbols used to represent deflections.

A deflection is between two directions, and a direction between two points, and, therefore, a deflection can be represented only by means of four letters, while an angle is represented by three.

There are many who think that a book of mathematical

problems should resemble a collection of recipes for cooking, in which the recipe for each dish stands on its own merits, and is independent of what may precede or come after.

To such it may be stated that, as the preliminary step in the old recipe for cooking a hare is first to catch the hare, so the first step in the computation of mathematical quantities is to grasp the quantities, after which the computation will be comparatively easy.

It is to the explanation of the natures of the quantities computed that a large portion of this treatise is devoted, and to those who start in the middle of the book the formulæ and solutions will probably be unintelligible.

If, however, the natures of the quantities, and the symbols by which they are represented, are first studied and understood, no great difficulty will be found in the practical application of the greater portion of the formulæ, and, although some of the results from general formulæ may appear more curious than useful, they serve, at least, to illustrate the general principles on which they are based.

From lack of time and patience for their preparation the tables of sines, cosines, versed sines, radii, etc., usually appended to works on railway surveying, have been omitted in this treatise. They are, however, absolutely necessary in making the computations, and should include a table of natural and logarithmic versed sines. Those contained in Searles' treatise on railway surveying will be found to be specially adapted for the purpose.

ISAAC W. SMITH.

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CHAPTER I.

INTRODUCTION.

1. BETWEEN the practical and theoretical engineer, so called, there is this distinction: the one can understand the natures of quantities which he cannot compute, and the other can compute quantities which he cannot understand.

To these two classes add a third and larger class who have neither practice nor theory, and there will be included a very considerable portion of those engaged in the practice of civil engineering, and especially in that branch, if it may be so called, engaged in the conduct of railway surveys.

In such surveys the practical man understands what is needed, and accomplishes it imperfectly by dint of the process known as fudging, while the theorist makes exact measurements and calculations which have no connection whatever with the end in view.

2. *Mathematics.*—The object of every mathematical computation is to determine the ratio, or relative magnitude, of two quantities of the same kind.

In pure mathematics, consisting of arithmetic and algebra, ratio, whether represented by numbers or letters, is the only quantity considered. In the application of mathematics to the measurement of quantities other than ratios, the ratio between two quantities of one kind is calculated generally as a part, or multiple, of the ratios between quantities of other kinds; but the addition, subtraction, multiplication, or division, is of the ratios, and not of the quantities themselves, and the mere manipulation of the numbers is without reason or practical result, unless the nature and magnitude of the

standard, or unit of measurement, to which each ratio or number refers, is understood and borne in mind.

Of mathematics it has been said, that it is a mill from which nothing can be ground out which has not been put in, and to the misapprehension of the quantities to which, in numerical calculations, the ratios refer, is due what a railroad statistician has termed the immense latent power of lying inherent in figures.

3. *Definitions of Quantities.*—It is claimed by authorities in geometric science that the conclusions are based upon definitions easily understood, and on self-evident axioms and postulates; but, leaving out of question their unintelligible definitions of a point as that which has position without magnitude, and of a line as that which has length without breadth, it will be found, not only that they differ materially in their definitions of quantities, but that their definitions are, in some points, in direct opposition to their conclusions. Euclid, as translated by Playfair, defines a plane rectilineal angle as "the inclination of two straight lines which meet and are not in the same straight line."

According to Legendre, "when two straight lines meet, the quantity, whether greater or less, by which they depart from each other as to their positions, is called an angle."

According to Davies, late professor of mathematics in the United States Military Academy, "an angle is the space between two straight lines produced to intersection."

Legendre does not explicitly state that the sides of an angle cannot be parts of the same straight line, but, from his definition of a straight line, he deduces the proposition that "two straight lines which have two points common coincide throughout and form one and the same straight line"; and, as two lines which are parts of the same straight line cannot depart from each other, they cannot contain any angle between them.

From these and other definitions which might be cited, it will be found that, however much geometers may differ amongst themselves as to the nature of an angle, they agree with Euclid in considering it as a relation between two straight lines which meet and *are not in the same straight line;* and, in fact, it is sufficiently evident that, under any possible meaning which can be attached to the terms, there can be no inclina-

tion, departure, or space between two straight lines which are parts of the same straight line.

And yet by Euclid, and others who have written since his time, the conclusion based upon this definition is, that when the angle between two straight lines is equal to two right angles, the two sides are one and the same straight line.

The chain of reasoning, so called, by which a conclusion so directly opposed to the premises is reached, will be seen by the following quotations from Legendre :

Definition.—"When two straight lines, BA, BC, (figure 1), meet, the quantity, whether greater or less, by which they depart from each other as to their position, is called an angle ; the point of meeting, or intersection, B, is the vertex of the angle ; the lines BA, BC are its sides."

"Angles, like other quantities, are susceptible of addition, subtraction, multiplication, and division ; thus, the angle CBD (figure 1) is the sum of the angles ABC and ABD, and the angle ABC is the difference between the angles CBD and ABD."

Theorem.—"Two straight lines which have two points common coincide throughout, and form one and the same straight line."

Theorem.—"If two adjacent angles, ABC, ABD, (figure 1), are together equal to two right angles, the two exterior sides, CB, BD, are in the same straight line."

It therefore follows that the quantity, (two right angles), by which the two straight lines, BC and BD, depart from each other, is twice as great when they are one and the same straight line as when they are perpendicular to each other.

It is further to be observed that in no instance is an angle of 180 degrees represented as between two straight lines, unless by the intervention of a third line ; and also that the definitions are indefinite, because there are four angles between any two straight lines, and four different lines making the same angle with any straight line AG, figure 4.

If, however, an angle is considered as a deflection, or turn, from one direction to another, and the distinction made between the direction BC and the opposite direction, CB, there will be no difficulty in the proposition that, when the deflection from BC to BD is equal to 180 degrees, the points B, C, and D are on one and the same straight line.

4. *Plane trigonometry*, or at least the problem most frequently recurring in plane trigonometry, is the determination of the relative positions of points, and the position of a point can be determined only by means of its distance and direction from some other point, or from two straight lines or axes.

As, therefore, the solution of almost every problem involves the determination of direction, it is singular that the subject of direction is seldom or never taken into consideration, or even alluded to, in trigonometric text-books.

In the application of trigonometry to plane surveying and navigation the distinction between north and south is necessarily recognized, and angles are considered as deflections to right or left, or to east or west, from a north direction.

In all works on plane surveying there is, however, the same discordance between premises and conclusions as in geometric text-books. In a work, for instance, on the elements of plane surveying and navigation, by the late Mr. Davies, professor of mathematics at the United States Military Academy at West Point, an angle is defined as "the portion of a plane between two straight lines which meet at a common point"; a course, as "a line traced or measured on the ground"; and the bearing of a course, as "the angle which it makes to east or west with a north direction or south direction."

To determine the angle between any two courses from their bearings, four rules are given, occupying nearly a page in print, and difficult to remember or apply. In a remark appended to the rules it is then stated that "the above principles are determined under the supposition that the two courses are both run from the same angular point; hence if it is required to apply them to two courses run in the ordinary way, as we go around the field, the bearing of one of them must be reversed before the calculation for the angle is made." In illustration of these rules it is stated that, when the bearings of two courses are N. 39° W. and S. 48° W., the angle between them is 93° when measured from the same angular point, and 87° as "we go around the field."

The deflection from any one direction to another is, by proposition V., the algebraic difference of their bearings from any third direction, and, calling right deflections positive, the deflection from a direction bearing N. 39° W., or $-39°$, to

a direction S. 48° W., or —132°, is (— 132°) — (— 39°) = — 93°, and the deflection to a direction bearing N. 48° E. would be + 87°. Now it is true that angles of 87° and 93° are supplements one of the other, and that the sines and departures for these angles may be the same, but the cosines and latitudes are different in sign. On the surveys of railroad alignments angles are seldom measured from *the same angular point*, but generally *as we go around the field;* and as the deflection, or angle, from one tangential direction to another, is equal to the algebraic sum of the intermediate deflections, the substitution of the supplement for the angle would, in this case, result in an error of 6 degrees.

It may be objected to these remarks that the surveys of curvilinear alignments of railroad-tracks are special applications, requiring special rules; but such is not the case. In applied mechanics, for instance, forces are represented, in magnitudes and *directions*, by the sides of polygons, open or closed, and the magnitude and direction of the resultant force are determined by precisely the same principles which are applied, in problem XI. of this treatise, to the determination of the length and bearing of the closing line to a series of lines given in length and direction.

In railroad surveys it is necessary always to determine the direction and magnitude of the deflection from one tangent-point to another, and this deflection has been unfortunately termed the angle of intersection, although it is seldom convenient, and never necessary, to produce the tangents to intersection in order to measure the angle between them.

5. *Sines, cosines, etc.*—With regard to other quantities there is not the same radical error of definition as in the case of angles, but there is a great lack of precision in the text-books in general use.

A ship, for instance, sails N. 30° E. 100 miles; the departure is to the east, and 100 times the sine of 30°, or 50 miles; the latitude is to the north, and 100 times the cosine of 30°, or 86.6 miles; the sine of 30° is therefore the rate of departure, or the departure in one mile, from a north line, and the signs of the sine and of the bearing are the same; that is to say, the departure and sine will be to the east when the bearing is to the east. The cosine of 30° is the rate of departure from a west line, or the latitude in a distance of one mile, and has

the same sign with the bearing from a west line, which, in this case, would be 120° to the right, or north.

But, turning to Legendre, it will be found that "the sine of an arc is the perpendicular let fall from one extremity of the arc upon the radius which passes through the other extremity." "The cosine of an arc is the sine of the complement of the arc, and is equal to that part of the radius comprehended between the centre and the foot of the sine."

According to this definition, the sine or cosine of an angle varies with the length of the radius; but as these quantities are constant for the same angle, the definition is not correct.

If it was intended to convey the idea that the sine is the ratio between the radius and perpendicular, or between the hypothenuse and perpendicular of a right-angled triangle, the idea is certainly loosely expressed for a mathematical text-book.

By means of circles the relative magnitudes of the sines of different angles may be graphically represented, but nothing can be deduced with regard to the signs of the quantities for angles of different magnitude.

In plane surveying and navigation, the problem most frequently recurring is to determine the relative positions of points, or the distance and direction from one point to another, and hence has arisen the plain and simple theory of latitudes and departures, in which the latitude and departure of the closing line is calculated as the algebraic sum of the latitudes and departures of the several lines between its extremities.

On this proposition, (XIII), and on proposition IV, that the total deflection is the algebraic sum of the partial deflections, are based all of the problems in this book.

The general problem presented is, to make a deflection, or turn, of given magnitude, over two circular curves connected by a common tangent, (which may be a point or line), or intersecting at a given angle. The quantities which enter into the equations are: the total deflection, I; the deflections Ia and Ib over the circular curves; the angles A′ and B′ which the line (**AB**), between the tangent-points, makes with the tangents; the distance, t, from A to B; the radii, a and b, of the two curves; the length, t, of the common tangent, or the angle, F, at intersection, for intersecting curves.

For the solution of the general problem, the line from the centre of one curve to the centre of the other is considered as the closing line to the radii, a and b and the distance, i, between the tangent-points, and the bearings of certain radial directions are calculated ; the required tangential deflections are then determined as the algebraic sums or differences of the given and calculated bearings. Problems XIX. to XXVII. are special cases of the same problem, and all are derived by the application of the same rules, which are invariable in their application, and give, not only the magnitudes, but the signs of the deflections for the different curves shown in figures 50 to 119.

When possible, the formulæ for special cases have been embodied in tables, with reference to particular figures, in which the proper signs are given to the terms for the different forms of curves represented in the figures.

In all of these formulæ the same symbols are used, which represent the same quantities throughout the book; and unless the reader will take the trouble to become familiar with the symbols and the quantities which they represent, the problems, like the figures, will partake of the nature of a Chinese puzzle, requiring a special study for the understanding of each one.

6. *Measurement of Deflections.*—As deflections are measured by the rotation of a rigid body on its axis, it is necessary to a proper understanding of the principles of deflections to consider the subject of motion.

Motion of Points.—A point moves when it changes its position, either in distance or direction, in relation to some other point; and, so long as the distance and direction from one point to another remain unchanged, they are at rest in relation to each other, although they may move relatively to other points. Motion, like magnitude, is therefore relative, and not absolute.

Motion of Rigid Bodies.—A rigid body may be considered as composed of points which have no motion with relation to each other, or in which the relations of distance and inclination between the points remain unchanged during any motion to which the body may be subjected. Such a body moves when any one of its points changes position, whether in direction or distance, in relation to any point not connected with the body ;

and there are two classes of motion, one of translation, and the other of rotation.

Motion of Translation.—When all the points of the body move with equal velocity, and in the same direction, the directions between the several points of the body remain unchanged, and the motion is one of translation, the body being simply translated from one position to another.

Motion of Rotation.—When any one point of a body moves in a direction, or with a velocity, different from that of any other point of the body, the direction between the points changes, and the body has a motion of rotation. The motion of rotation may be combined with a motion of translation, but the two are entirely independent, and one may be changed or stopped without affecting the other. The paths described by the points may be straight or curved, and the direction of the motion of translation may be changed or reversed ; but so long as the directions between the points do not change, the motion is one of translation only, like the motion of a car, back and forth, on a straight railroad-track. The motion of the earth in its orbit is one of translation from east to west, and variable. The motion of rotation on its axis is one of rotation from west to east, and uniform.

Axes and Planes of Rotation.—Although the unequal motion of any two points in a body is an indication of a motion of rotation, there will be, in every rotating body, certain points which move in like directions and with equal velocities, and the directions between such points remain unchanged. The lines through such points are called *axes of rotation*, as, for instance, the axis of the earth and all lines parallel to the axis.

Should the body turn, without motion of translation, the axes would move on the surfaces of cylinders; lines inclined more or less than 90 degrees to the axes, on the surfaces of cones, and lines at right angles to the axes in planes, called *planes of rotation*.

All planes at right angles to the axes of rotation are planes of rotation, as, for instance, the equatorial plane of the earth, and the horizontal plane of a theodolite when the axis is vertical.

Complete Rotation of a Turning Body.—A body makes a complete rotation when any line inclined to the axis, or any

plane inclined to the plane of rotation, turns from any fixed direction back to the same direction again. A meridian plane, for instance, through the axis of the earth, and perpendicular to the equatorial plane, turns from the direction of a fixed star back to the same direction again once in twenty-four sidereal hours.

The orbit of the earth and a fixed star are, in relation to each other, points; and as the earth has, therefore, no motion of translation in relation to the star, the rotation of the meridian plane from a fixed star back to the same star again is always made in twenty-four sidereal hours. The earth has, however, a sensible motion of translation from east to west, in relation to the sun, and moves from under the sun while turning from west to east, so that it makes more than a complete rotation while the meridian turns from the sun back to the sun again, and the solar day is therefore longer than the sidereal day.

A plane deflection is the change in the direction from one point to another in the plane of rotation of a turning body, and is at the rate of 360 degrees for each complete rotation of such body; and the deflection from a fixed direction, AB, to another fixed direction, CD, is the change of direction of such points while turning from the direction AB to the direction CD. The magnitude of this deflection is independent of any motion of translation to which the body may be subjected, and will be the same, whether the lines connecting the points are produced to intersection or not, provided that they are in the plane of rotation of the body. Suppose, for instance, that AB and CD (figure 3) are directions between fixed points in the plane of the paper, which is also the plane of rotation of a turning body, and that the body, like the earth, makes a complete rotation in twenty-four hours, so that the direction from any one point to another in the plane of rotation changes at the rate of 15 degrees per hour.

If, in one hour, a point, P, in the plane of rotation, should be translated over any path, straight or curved, and the direction, PR, from that point to another point, R, also in the plane of rotation, should change from the fixed direction AB to the fixed direction CD, the deflection AB.CD, from AB to CD, would be equal to 15 degrees.

The general relation between the lines **AB** and **CD**, which may be called the inclination, exists without reference to

motion ; but the *quantity measured* is not the inclination, but the deflection, and this can be measured only by means of a motion of rotation.

Positive and Negative Deflections.—A direction may be defined as through two fixed points, as C and D ; it may also be defined by its deflection, or bearing, from some fixed direction, as AB, but, in this case, both the magnitude and the direction, right or left, of the bearing must be given, because, for instance, a direction bearing 30 degrees to the right of north is different from a direction bearing 30 degrees to the left.

Right and left deflections are, therefore, to be distinguished as positive and negative quantities. The direction of rotation, right or left, should, however, be distinguished from the direction of the motion of translation, as a body may be translated right or left, or back or forth, without rotation.

Although a direction may be thus defined by the magnitude and sign of its deflection, or bearing, from some other direction, the deflections measured from one direction to another will not be always the same, for the changing direction, by which the deflections are measured, returns always to the same direction again after each complete rotation, right or left, and hence the deflections measured between the same directions may differ by any whole multiple of 360 degrees ; and, conversely, if the algebraic difference of the deflections to AB and CD, from any third deflection, SN, is equal to any whole multiple of 360 degrees, AB and CD are one and the same direction, as exemplified under the head of equivalent deflections in propositions II. and VII., and problem II.

From these propositions, and from the proposition (IV.) that the deflection from one direction to another is equal to the algebraic sum of the partial deflections made in turning, whatever may be the positions of the deflection points, may be derived every other proposition having reference to deflections only, and not to other quantities.

Instrumental Measurement of Deflections.—It may appear unnecessary to complicate the definition of an angle by introducing the idea of motion, but the nature of the quantity, the mode of measurement, and the adjustments of the instruments by which the measurements are made, cannot otherwise be properly understood.

In plane surveying it is assumed that vertical lines from

the centre of the earth are parallel, or that the surface is a plane—an assumption that will not lead to appreciable error in the class of sufveys under consideration.

As the deflection points, and the lines connecting them, are usually not in the same plane of rotation, the deflections are measured between their vertical projections on a horizontal plane.

The process of measurement may be illustrated by the motion of the vertical side plane of a door turning on a vertical axis.

All lines seen along that plane would be vertically projected on the horizontal line of the lower edge, and that line in turning would measure the deflections between the vertical projections of the inclined lines seen along the plane of the door, and not between the lines themselves. The vertical rotating plane of a magnetic compass is the plane passing through the vertical line of the sights and rotating around the fixed line of the needle. The bearings of the vertical projections of the inclined lines seen through the sights are measured from the fixed direction of magnetic north by means of a graduated circular arc, which turns around the needle, and the deflections from one direction to another may be calculated as the algebraic difference of their bearings taken at any points on the lines or at their intersections. The vertical plane of a transit or theodolite is the plane of the line of collimation, or of the line of sight of the telescope, which is supposed to turn on a horizontal axis.

The deflections between the vertical projections of the lines seen through the telescope are measured by means of two horizontal graduated arcs, one of which is fixed, while the other turns with the vertical plane.

The angles measured on the horizontal plates, or planes of rotation, are really the rotations of the vertical plane in which the line of sight of the telescope revolves ; and this plane may be supposed to move, at the deflection points, by a motion of rotation from one direction to another, and by a motion of translation between the points, the deflection, from the initial to the terminal direction, being equal to the algebraic sum of the intermediate deflections, and independent of the relative positions of the deflection points. These deflections are measured by means of circular arcs, but they are not arcs

more than they are sectors of circles. It is on the same principle, and by the rotation of meridian planes of the earth, that the deflections from the direction of one star to the direction of another are measured on the equatorial plane, as a plane of rotation ; but in this case the deflection is measured by means of the intervals of time elapsing between the meridian passages of the stars, the quantity measured being the same in each case, and neither area, length, or time.

CHAPTER II.

PROBLEMS.

DEFINITIONS OF SYMBOLS.

Positive and Negative Signs.—Positive quantities are generally defined as additive, and negative quantities as subtractive, and indicated by the signs of addition and subtraction; but the definitions are erroneous, because a positive quantity may be subtracted, and a negative quantity added.

To obviate the confusion arising from the use of the same sign with different significations, when a plus or minus sign is used to indicate whether a quantity is positive or negative, it will be placed, either above the quantity, or prefixed (within brackets). Where, however, there is no liability of mistaking the signification, it will be prefixed in the usual manner.

Thus, for instance, $a = b - c$ indicates that a is equal to the algebraic difference between b and c, whether they are positive or negative, but $a = (+b) - (-c)$ is the value of a when b is positive and c negative. The first equation expresses a general relation for all values of b and c, and the second the arithmetic value ($a = b + c$), when c is negative.

In all algebraic and trigonometric equations, the sums and differences are algebraic, and to obtain the arithmetic sum or difference, the sign prefixed to a negative quantity must be changed, from plus to minus, or from minus to plus.

The required arithmetic sum or difference will then be the difference of the sums of the plus and minus quantities; positive when the sum of the plus quantities is the greatest, and negative when it is least.

Points are represented by the letters A, B, C, etc., and the same symbols are used to represent the numerical values of angles or deflections.

DIRECTIONS.—AB is the direction from A to B, and BA the opposite direction, from B to A.

Directions of Co-ordinate Axes.—SN and EW are the directions of a pair of rectangular co-ordinate axes. If SN were north, EW would be west, and, in any case, it is to be understood that the deflection from EW to SN is 90 degrees to the right.

DEFLECTIONS, or ANGLES, are from one direction to another, and if deflections in one direction, as to the right, are assumed to be positive, deflections to the left will be negative.

AB.CD is the deflection from AB to CD; CD.AB = (— AB.CD) is the deflection from CD to AB, and BA.CD = —(180° — AB.CD) is the deflection to CD from a direction BA, opposite to AB.

Thus, for instance, if the deflection from AB to CD is 60° to the right, the deflection from CD to AB is 60° to the left, and the deflection from BA to CD is (180° — 60° = 120°) to the left.

BEARINGS.—When deflections to different directions are from the same axis, they are called bearings.

Thus, for instance, SN.AB denotes the bearing from SN to AB, and (AB.CD = SN.CD — SN.AB) (see proposition V.) expresses the proposition that the deflection from one direction to another is equal to the algebraic difference of their bearings from any third direction, SN.

Numerical Values of Angles are expressed by the letters A, B, C, etc. The same symbols are used for points, the context showing the sense in which the letters are to be understood.

SYMBOLS FOR STRAIGHT LINES.—**AB** represents a straight line extending indefinitely through A and B, in connection with the direction from A to B. **SN** is an axis, with reference to the direction from S to N. Thus, for instance, if a point is said to be 50 feet to the right of **SN**, it is to be understood that it is to the right looking from S to N, which would be to the left looking from N to S.

In the same manner, if SN is north, east is to the right of **SN**, and to the left of **NS**, and the direction NS is south.

DISTANCES.—If A and B are points on a straight line, **(AB)** is the distance from A to B in the direction AB, and **(BA)** the distance in the opposite direction, BA.

If distances measured in the direction AB are considered positive, distances in the opposite direction, BA, will be negative.

Thus, for instance, the equation $[(\mathbf{BC}) = (\mathbf{AC}) - (\mathbf{AB})]$ is the expression of the proposition [XI. (2)] that the distance from one point to another is the algebraic difference of their distances from any third point on the same straight line.

The same relation may be expressed by the equation $[(\mathbf{AB}) + (\mathbf{BC}) = (\mathbf{AC})]$, and if it be assumed that distances between parallel lines are equal when measured in the same direction, the whole of plane linear geometry may be based upon the propositions expressed by the equations:

Prop. IV. (1). $AB.CD = AB.BC + BC.CD.$
Prop. XI. (1). $(\mathbf{AC}) = (\mathbf{AB}) + (\mathbf{BC}).$

The Numerical Values of Distances, or their ratios to given units of distance, are expressed by the letters a, b, c, etc.

PROBLEM I.

TO REDUCE MINUTES AND SECONDS TO DECIMALS OF A DEGREE, AND DECIMALS OF A DEGREE TO MINUTES AND SECONDS.

To reduce to decimals of a degree, move the decimal point of the seconds one space to the *left*, and divide by 6; the quotient will be decimals of a minute; affix these decimals to the minutes; move the decimal point one space to the *left*, and divide by 6; the quotient will be decimals of a degree.

To reduce decimals of a degree to minutes and seconds, move the decimal point one space to the *right*, and multiply by 6; the product will be minutes and decimals; move the decimal point of the decimals of a minute one space to the *right*, and multiply by 6; the product will be seconds and decimals.

Example 1. Reduce 18° 1′ 24″ to degrees and decimals.

$24'' = 2.4 \div 6 = 0.4'$. $1'\ 24'' = 1.4' = .14 \div 6 = .0233°$.
$18°\ 1'\ 24'' = 18.0233°$.

Example 2. Reduce 18.0233° to degrees, minutes, and seconds.

$.0233° = .233 \times 6 = 1.398'$. $0.398' = 3.98 \times 6 = 24''$.
18.0233 degrees is therefore equal to 18° 1′ 24″.

In railroad surveys, where minutes and seconds are to be multiplied or divided, it will be found convenient to reduce them to decimals of a degree by the above method. The greatest liability to error is in the placing of the decimal point.

PROBLEM II.

EQUIVALENT DEFLECTIONS—PROPOSITIONS II. AND VIII.

Given a deflection greater than 180 *degrees; required the equivalent deflection less than* 180 *degrees.*

Let L be the lesser, and M the greater deflection.

When the greater deflection is less than 360 *degrees,* L will be equal to (360°—M), with a sign different from that of M.

Examples.—In these examples a plus sign will indicate a deflection to the right, and a minus sign a deflection to the left.

If M is +200°, or 200 degrees to the right, L = —(360°—200°) = —160°, or 160 degrees to the left.

M = —181°; L = +(360°—181°) = +179°. M = +180°; L = —(360°—180°) = —180°.

When M *is greater than* 360 *degrees,* L is equal to the algebraic sum of M and of the whole multiple of 360 degrees, nearest to M, and with a contrary sign.

M = +1862° = (5 × 360° + 62°). L = +1862° − 1800° = +62°.
M = −380° = −(360° + 20°). L = −380° + 360° = −20°.
M = −560° = −(720° − 160°). L = −560° + 720° = +160°.

From these rules, and from proposition VIII., it follows, if the algebraic difference of the deflections from any direction, SN, to two directions, AB and CD, is equal to any whole multiple of 360 degrees, that AB and CD are one and the same direction; the deflections are equivalent, and one may be substituted for the other in any trigonometric formula. It follows also that, in any such formula, the sign prefixed to 180° may be always changed from plus to minus, or from minus to plus. Thus, for instance, −180° + A = −(180° − A) may be substituted for +180° + A, because −180° + A − (+180° − A) = −360°, but the supplement of an angle cannot be substituted

for the angle, nor is it allowable to change the sign of A in the above examples.

Thus, for instance, $(-180°+20°)=-160°$ may be substituted for $(+180°+20°)=+200°$, but not for $(+180°-20°)=+160°$, because, in the latter case, the algebraic difference, $(-160°)-(+160°)=-320°$, is not a whole multiple of 360°.

PROBLEM III.

PROPOSITIONS IV. AND V.

Given the bearing from an axis SN *to a direction* AB, *and the deflections,* AB.CD, CD.EF, EF.GH, ——, UV.WX, *and* WX.YZ, *from* AB, *to and from each direction between* AB *and* YZ; *required the bearing, from* SN, *to each direction, and the deflection from any one direction to another.*

The bearing, from SN to any direction YZ, is the algebraic sum of the deflections between SN and YZ.

The deflection from any one direction to a second is the algebraic difference, the bearing of the second minus the bearing of the first.

Formulæ.

$$SN.YZ = SN.AB + AB.CD + CD.EF, \text{ —— } + UV.WX + WX.YZ. \quad (1)$$

$$GH.CD = SN.CD - SN.GH. \qquad CD.GH = SN.GH - SN.CD. \quad (2)$$

In (1), the deflections are to be measured, first *to,* and then *from,* each direction between SN and YZ.

In (2), if the deflection *from* GH to CD is to be calculated, the bearing of the *initial* direction, GH, is to be the subtractive term of the difference in the second member of the equation.

In (1) and (2), deflections in one direction, as to the right, are to be considered positive, and those in the opposite direction negative, and the sums and differences are to be understood in the *algebraic sense;* that is to say, the sign prefixed to a negative term must be changed from plus to minus, or from minus to plus, and the bearing, or deflection, in the first member of the equation, will then be the *arithmetic* difference of the

sums of the plus and minus terms in the second member, *with the sign of the greater sum.* If the calculated bearing, or deflection, is greater than 180 degrees, it should be reduced to an equivalent deflection less than 180 degrees by the rules of problem II.

It is to be noted, also, that the bearings are all to be estimated *from one axis*, and not, as in plane surveying, by *quadrants*, some from north and others from south, and if SN is assumed to be north, bearings from south are to be reduced to bearings from north by the following formula:

$$SN.AB = 180° - NS.AB. \quad (3)$$

Thus, for instance, calling the right positive, S. 15° E. is N. 165° E., or +165°, and S. 15° W. is N. 165° W., or −165°.

In all of the following examples, the right is assumed to be positive, and the left negative, and, when reference is made to north, the east will, therefore, be positive, and the west negative.

Example 1 (figure 5). AB.CD = +50°; CD.EF = +70°; EF.GH = +60°; required the bearing of each direction from AB.

AB.CD is 50 degrees to the right, = +50°.
AB.EF = AB.CD + CD.EF = +(+50°) + (+70°) = +120°.
AB.GH = AB.EF + EF.GH = +(+120°) + (+60°) = +180°.

In figure 5, the deflection points coincide at the point O.

In figure 5*a*, the figure is a three-sided open polygon.

In figure 5*b*, the line **AB** coincides with the line **GH**, and the figure is a triangle; but in the three figures the relations between the directions are the same, and the angle between AB and GH is equal to 180° whether the lines **AB** and **GH** coincide, or are parallel.

In figure 5*b*, the three angles, being each less than 180° and in the same direction (to the right), are the interior angles of the triangle. These angles, measured to the left, would be AB.CD = −310°; CD.EF = −290°, and EF.GH = −300°, and the deflection from AB to GH, equal to the sum of these deflections, would be −900°. In this case the deflection actually made in turning from AB to GH would be 900 degrees, which, by problem II., is equivalent to (−900° + 720° = −180°), or to (+1080° − 900° = +180°).

In the same manner it may be shown, whatever may be the number and magnitude of the deflections made in turning from a direction AB to a direction, BA, opposite to AB, that their algebraic sum will be equal to 180 degrees, or to 180 degrees, plus or minus a whole multiple of 360 degrees.

Example 2 (figure 6). In this figure the initial direction is from a point A, on the circumference of a circle; the deflection points are the intersections of the sides of hexagons on the circumferences of two circles reversing at E, and the terminal direction, GT, is that of a tangent from the point G. The deflections, however, are independent of the positions of the deflection points, and the results would be the same should all coincide in one. The bearings from SN, considered as north, to each direction, are calculated in the following table:

Directions.	Deflections.	Bearings.	
		From AN.	By Quadrant.
AN		0	North.
AB	AN.AB = + 30°	+ 30°	N. 30° E.
BC	AB.BC = + 60°	+ 90°	East.
CD	BC.CD = + 60°	+ 150°	S. 30° E.
DE	CD.DE = + 60°	+ 210° / − 150°	S. 30° W.
EF	DE.EF = 0	− 150°	S. 30° W.
FG	EF.FG = − 60°	− 210° / + 150°	S. 30° E.
GT	FG.GT = − 30°	+ 120°	S. 60° E.
	AN.GT = + 120°		

In this example, the given bearing of AB is zero, and the bearing of any other direction is the algebraic sum of the bearing *to* that direction, in the second column, and of the preceding bearing.

When the bearing, thus calculated, exceeds 180°, substitute the equivalent bearing obtained by subtracting from 360° and changing the sign, as in the case of the bearings of DE and FG.

The bearing, + 120°, from the initial to the terminal direc-

tion, is equal to the algebraic sum of the deflections in the second column—that is to say, to the arithmetic difference of the sums of the plus and minus terms, with the sign of the greater sum.

The deflection from any one direction to another may be calculated by formula (2) in the following manner:

$$BC.EF = \begin{pmatrix} AN.EF - AN.BC \\ -150° - (+90°) \end{pmatrix} = -240°, \text{ equivalent to } +120°.$$

$$AB.EF = \begin{pmatrix} AN.EF - AN.AB \\ -150° - (+30°) \end{pmatrix} = -180°, \text{ equivalent to } +180°.$$

As the deflection from AB to EF is equal to 180°, the lines **AB** and **EF** are therefore parallel.

Example 3 (figure 7). **AI** and **BI** are tangents—that is to say, they are tangent to circles through the tangent points A and B.

The circles are not shown, but may be seen in any of the succeeding figures representing the connection of tangents by two curves with a common intermediate tangent, or intersecting at an angle, F, called, in railroad surveys, the frog angle.

The deflection over one or more curves is called the angle of intersection, although it is seldom that the tangents are produced to intersection, or that the angle can be measured at the intersection point, I.

Throughout this treatise this angle of intersection is represented by the letter I, and it is proposed to explain the manner in which it may be determined without producing the tangents to intersection.

Take any two convenient points, A and B, on the tangents, and measure the acute angles, A′ and B′, between the line **AB** and the tangent lines **AI** and **IB.** It will not be necessary to note the signs of these angles, but they are always to be acute, or less than 90 degrees, and, as a rule, A′ is to be the greater of the two.

The two tangents, with the line **AB,** will form a triangle, which will be as represented in figure 7, or as in figure 7*a.*

In figure 7, the acute angles are equal to the interior angles adjacent to the base **AB.** In figure 7*a,* the greater angle, A′, is exterior to the triangle, and the adjacent interior angle is obtuse.

In figure 7, the angle of intersection, when not exceeding 180 degrees, will be equal either to (A′+B′), or to 180°—(A′+B′).

In figure 7*a*, it will be either (A′—B′), or 180°—(A′—B′), as shown by the following equations:

Fig. 7. $AI.IB = \begin{pmatrix} AB.IB - AB.AI \\ (+B') - (-A') \end{pmatrix} = +(A'+B').$

$$AI.BI = \begin{pmatrix} AB.BI - AB.AI \\ -(180° - B') - (-A') \end{pmatrix} = -[180° - (A'+B')].$$

Fig. 7*a*. $IA.IB = \begin{pmatrix} AB.IB - AB.IA \\ (-B') - (-A') \end{pmatrix} = +(A'-B').$

$$IA.BI = \begin{pmatrix} AB.BI - AB.IA \\ +180° - B' - (-A') \end{pmatrix} = +180° + (A'-B') = -[180° - (A'-B')].$$

In figure 7, the deflections are made from AI, to IB in one case, and to the opposite direction, BI, in the other, and hence they are different in sign and magnitude. In figure 7*a*, the deflections are to IB, in one case, and to the opposite direction, BI, in the other.

It will not be necessary to take into consideration the sign of I, but the following rule should be thoroughly understood and recollected, as it is used in nearly every problem in the book:

The two tangents intersecting at I, and the line **AB**, will form a triangle, which will be similar either to figure 7 or to 7*a*.

I *will be equal to* (A′+B′) *or to* 180°—(A′+B′), *when the triangle is similar to figure* 7, *and to* (A′—B′) *or to* 180°—(A′—B′), *when it is similar to figure* 7*a*.

It is assumed that the deflection will be made so as not to exceed 180 degrees. For forms of curves for particular values of I, see problem V.

Example 4. Let TN be the true, and MN′ the magnetic, north; TN.MN′ = V, the variation, or deflection from true to magnetic north, and MN′.AB = B a magnetic bearing, or deflection from magnetic north to a direction AB.

Given the variation, and the magnetic bearing of AB; *required the true bearing,* TN.AB.

As the variation, V, is the deflection from true to magnetic

north, and B is the deflection from magnetic north to AB, the bearing of AB from true north will be as follows:

$$TN.AB = TN.MN' + MN'.AB = \text{algebraic sum, } (V + B).$$

In the following examples, east variations and bearings will be positive, and west negative:

If V is 21° east, or +21°, and B, S. 10° W., or −170°.
TN.AB = +21°+(−170°)= −149°, or S. 31° W.

If V is 16° west, or −16°, and B, S. 80° E., or + 100°.
TN.AB = −16°+100°= +84°, or N. 84 E.

If V is 21° east, or +21°, and B, N. 10° W., or −10°.
TN.AB = +21°+(−10°)= +11°, or N. 11° E.

Example 5 (figure 8). Two lines, **AI** and **IB**, intersect at an inaccessible point, I, and in order to determine the deflection from the direction AI to the direction IB, their bearings from a third direction, AB, are taken as follows:

$$AB.AI = -20°;\ AB.IB = +10°.$$

The deflection from one of these directions to the other is:

AI.IB = [Bearing, (+10°), of IB]−[Bearing, (−20°), of AI]
= +30°.

IB.AI = [Bearing, (−20°), of AI]−[Bearing, (+10°), of IB]
= −30°.

Care must be taken to measure both bearings from the same direction AB, not one from AB and the other from BA, which would result in an error of 180 degrees, and for the same reason, if the deflection AI.IB is to be measured, the bearings must be taken to AI and IB, and not to their opposite directions IA and BI.

The deflection from AB to IA, for instance, is +160°, and the deflection from IA to IB is, therefore, (+10°−(+160°)= −150°, or 150 degrees to the left, while the deflection from AI to IB is 30 degrees to the right. The bearings may, however, be taken from any one direction whatever, and to the right or left, and the calculated deflections will be the same or equivalent.

Example 6. The bearings of two directions are, (N. 39 W. = −39°), and (S. 48° W.= −132°), and the deflection from the

first to the second is, therefore, [—132°—(—39°)=—93°]. This example is from a work on surveying and navigation, by Mr. Davies, late professor of mathematics at the United States Military Academy at West Point, in which it is stated that the deflection would be 93°, if the bearings were taken from the same point, and 87° if taken "*as you go around a piece of land.*"

A ship sailing N. 39° W. would never head S. 48° W. by changing its course 87 degrees, and the change made in the inclination of two courses by going around a piece of land is difficult to understand.

PROBLEM IV.

PROPOSITION VI.

Given the bearings from two axes, AN *and* CN′, *to two directions,* AB *and* CD, *and the deflection from one axis to the other; required the magnitude and sign of the deflection from* AB *to* CD, *and from* CD *to* AB.

$$\left.\begin{aligned} AB.CD &= CN'.CD - AN.AB + AN.CN' = (-CD.AB) \\ CD.AB &= AN.AB - CN'.CD + CN'.AN = (-AB.CD) \end{aligned}\right\} \quad (1)$$

In this formula the sum and difference is algebraic, and if either of the deflections is negative the sign prefixed must be changed from plus to minus, or from minus to plus.

In calculating the deflection, AB.CD, the bearing, AN.AB, of the initial direction, AB, of the first member of the equation, must be the *subtractive* term of the algebraic difference of the bearings in the second, and the axis, AN, of AB, must be the initial direction of the deflection AN.CN′, between the two axes.

Example 1. TN and MN′ are the directions of true and magnetic north, and TN.MN′ the variation, or deflection from TN to MN′.

Given the variation, and true bearing of AB, *and the magnetic bearing of* CD; *required the sign and magnitude of the deflection from* AB *to* CD.

The true bearing, TN.AB, of AB, is N. 15° E., or +15°;

the magnetic bearing, MN'.CD, of CD, is S. 84° W., or −96°; the variation, TN.MN', from true to magnetic north, is +21°.

$$AB.CD = MN'.CD - TN.AB + TN.MN' = -96° - 15° + 21° = -90°.$$
$$(-96°) - (+15°) + (+21°)$$

With a variation of 21° west, or −21°, AB.CD = −96° − 15° + (−21°) = −132°.

The deflection from CD to AB is equal in magnitude to the deflection from AB to CD, but the signs are different, one being to the right and the other to the left.

Example 2 (figure 9). The intersection, I, of two lines, **BI** and **IC**, is inaccessible, and no convenient point can be found on one line which is visible from the other.

To determine the angle, BI.IC, a point S, visible from B and C, is therefore taken, and the following angles measured:

$$SB.BI = +86°;\ SB.SC = +70°;\ SC.IC = +79°.$$

The required deflection will therefore be as determined in the following equation:

$$BI.IC = SC.IC - SB.BI + SB.SC = +79° - (+86°) + (+70°) = +63°.$$

When the directions of the two axes are the same, the deflection AN.CN' is equal to zero, and the equation reduces to the form AB.CD=CN.CD−CN.AB, which is the same with (2) problem III.

PROBLEM V.

TANGENTIAL DEFLECTIONS, OR DEFLECTIONS OVER CIRCULAR ARCS.

(*For demonstration see proposition X.*)

A and B, figure 7, are points on tangent-lines intersecting at I; A' and B' are the greater and lesser acute angles between the tangent-lines and the line **AB.**

In figure 7, A' and B' are the interior angles at the base of the triangle AIB. In figure 7*a*, the greater angle A' is exterior to the triangle, and the adjacent interior angle is obtuse.

In figure 7, AI.IB = (A'+B'). AI.BI = 180° − (A'+B').
In figure 7*a*, AI.IB = 180° − (A'−B'). AI.BI = (A'−B').

I, called the angle of intersection, is the deflection from one tangential direction to the other, and, provided that it is measured so as not to exceed 180 degrees, it must always have one of the four values, $(A'+B')$, $(A'-B')$, $180°-(A'+B')$, or $180°-(A'-B')$.

This turn, or deflection, is to be made over two circular arcs connected by a common tangent, **T***a***T***b* (see figures 80 to 118), and I*a* is the tangential deflection from the initial tangential direction at A to T*a*T*b*, and I*b* the deflection from T*a*T*b* to the terminal tangential direction at B. The curves are reversed when, as in figure 90, the centres are on different sides of **T***a***T***b*, and compound when they are on the same side, as in figures 91 and 92.

The centres are on the same side of **AB** in figure 93, and on different sides in figure 94.

Given the angles A′ *and* B′, *and the following characteristics of the curves:* (I.), *whether the triangle* AIB *is similar to figure 7 or to figure 7a;* (II.), *whether the curves are reversed or compound;* (III.), *whether the centres are on the same or different sides of* **AB**; *required the value of* I *in terms of* A′ *and* B′.

The general arithmetic equation will be as follows:

$$I=(A'\pm B'),\text{ or }180°-(A'\pm B')=\pm Ia\pm Ib. \quad (1)$$

For tangents, as in figure 7, $I=(A'+B')$, or $180°-(A'+B')$.
For tangents, as in figure 7*a*, $I=(A'-B')$, or $180°-(A'-B')$.

The form of the triangle AIB will, therefore, determine whether I is expressed in terms of $(A'+B')$ or $(A'-B')$.

Whether I is equal to $(A'\pm B')$, or $180°-(A'\pm B')$, is determined by the following rules:

I *is equal either to* $(A'+B')$ *or to* $(A'-B')$;

For $\begin{pmatrix}\text{reversed curves with centres on different sides}\\ \text{compound curves with centres on the same side}\end{pmatrix}$ of **AB**.

I *is equal either to* $180°-(A'+B')$ *or to* $180°-(A'-B')$,

For $\begin{pmatrix}\text{reversed curves with centres on the same side}\\ \text{compound curves with centres on different sides}\end{pmatrix}$ of **AB**.

The signs of the deflections, whether positive or negative,

are to be determined in the manner explained in the succeeding article.

It will be seen, by reference to the figures, that any required deflection, $I = Ia + Ib$, may be made over several different forms of curves, subject to the limitations expressed in the rules above; but the value of I, and the signs of Ia and Ib, in the case of the curves in any particular figure, may be determined by inspection, in accordance with the rules of this problem.

In figure 114, for example, there are two pairs of connecting tangents to the same circles, and the curves are reversed with reference to one pair, and compound with reference to the other; the triangle AIB is similar to figure 7, and therefore I is equal either to $(A' + B')$ or to $180° - (A' + B')$; the centres are on the same side of **AB**, and therefore the algebraic sum, $Ia + Ib$, of the tangential deflections, between A and B and the extremities of common tangents, is equal to $(A' + B')$ when the curves are compound, and to $180° - (A' + B')$ when they are reversed, as calculated in example 1, problem XXV. In the greater portion of the problems of this book, the first step in the solutions is the determination of the values of I in terms of A′ and B′, and the examples in the succeeding article should, therefore, be carefully read and thoroughly understood.

Examples.—The deflection, AB.YZ, from the initial to the terminal direction of a series of directions, as explained in proposition IV. and problem III., is the algebraic sum of the deflections to and from each direction between AB and YZ, and the initial direction, and the direction, right or left, of the several deflections, may be represented graphically, so that there will be no difficulty in perceiving at a glance what must be the terminal direction, YZ, in order that the deflection, AB.YZ, may be equal to the algebraic sum of the deflections between AB and YZ. The lines **AZ** and **YZ** might be produced to intersection, and the angle between them measured, but there are four angles between each pair of intersecting lines, differing in sign or magnitude, and the deflection, in any particular case, can be determined only by the consideration of the partial deflections of which the total deflection is composed.

If the deflection is made over a series of circular arcs, touching at points, or lines, of compound or reversed curvature, the ection, from the initial to the terminal tangential direction

of the series, will be equal to the algebraic sum of the deflections over the several arcs composing the series; and conversely, if A and Z are the extreme tangent-points, and the tangent-lines through those points intersect at I, it may be determined by inspection of the figure, in any particular case, what are the signs of the partial deflections, and whether the total deflection over the series of arcs will be to the terminal direction IZ or ZI.

As preliminary to the consideration of the principles involved in the consideration of this problem, it is proposed to explain a method by which the direction (right or left) of the tangential deflections over each arc, the terminal tangential direction of the series, and the value of I in terms of A′ and B′, may be determined by inspection of the figure representing the curves. If the method is analogous to what is called the rule of thumb, it has, at least, the merit of clearness and simplicity, and, as it will be used in many of the succeeding problems, a greater space will be devoted to it than would otherwise be necessary.

Suppose the curves to be on the centre line of a railroad-track, and that a car moves over the track by a combined motion of rotation and translation. On the floor of the car, as a plane of rotation, let two points, P and R, be marked, one vertically over a point on the curve, and the other in such position that the direction PR may coincide with the tangential direction of the curve.

As the car moves over the track from A to B, PR will continue on the tangential direction at the point of curve vertically under P, and on arrival at B will take the direction of the terminal tangent of the arc AB.

The tangential deflection might be roughly measured by means of a compass at P, and the direction of the deflection would be right or left as the point R should appear to an observer at P to move to his right or left. The change of direction between every pair of points on the floor would be the same in sign and magnitude, and the sign of the deflection would therefore be the same, whether estimated looking from P to R, or from R to P, or from any one point in the plane of rotation to another; but, for convenience of illustration, PR will be supposed to be the tangential direction of the curve, and the signs of the deflections to be as they would appear looking from P towards R.

The car will move both by motion of rotation and translation, but the deflection is dependent only on the motion of rotation, and is not affected by the direction of the motion of translation, whether back or forth to the observer at P.

Should the car, for instance, move back and forth on a tangent, the direction PR would remain unchanged, and the deflection over any curve would be the same as though the car should turn on a turn-table at the intersection, I, of the initial and terminal tangents, from one tangential direction to another.

In the same manner, should a second car be placed at the centre of the curve, on a turn-table, and kept heading in the direction of the first car while moving over the track, the radial deflection would be the same in sign and magnitude with the tangential deflection. The tangential deflection, although equal to the radial deflection, may be best calculated as a property of the circle, of which neither the centre nor radius is a part, and, by the railroad surveyor at least, the circle should be considered as a curve of equal curvature—that is to say, as a curve on which the tangential deflections are directly proportional to the lengths of the arcs.

From this definition all the properties of the circle may be derived.

In the following examples the right will be considered positive, and the signs of the tangential deflections may be readily determined by the method explained above.

The deflection from the initial to the terminal direction of a series of arcs will be represented by the letter I; the partial deflections over the several arcs commencing at A, B, C, etc., by I*a*, I*b*, I*c*, etc., and the values of I, determined by the rules of problem V., will be given in terms of A′ and B′, as further exemplified in table III., problem XVI., and table XII., problem XXVI.

In figure 10, A, B, C, etc., are the initial, and A′, B′, C′, etc., the terminal points of the arcs, AA′, BB′, CC′, etc.

If the car should move over this series of curves, PR would change direction, successively, to the tangential directions of the different arcs, and the deflections would be as follows:

$$\underset{(+\,Ia)}{I = AI.A'B} + \underset{(-\,Ib)}{A'B.B'C} + \underset{(-\,Ic)}{B'C.DC'} + \underset{(-\,Id)}{DC'.D'E} + \underset{(+\,Ie)}{D'E.E'I}.$$

On arrival at C, the direction of the motion of translation would be reversed, and the car would move backwards, turning to the left from the direction B'C to the direction DC'; thence forward, and left, to the direction D'E, thence backwards, and right, to the direction of the terminal tangent E'I.

PR has therefore changed from the direction AI to the direction E'I, and the algebraic sum of the changes made in turning from the initial to the terminal tangential direction of each arc must, therefore, be equal either to AI.E'I = I, or to I plus or minus a whole multiple of 360 degrees, the arithmetic sum being as follows:

$$+\mathrm{I}a-\mathrm{I}b-\mathrm{I}c-\mathrm{I}d+\mathrm{I}e=\mathrm{AI.E'I}.$$

It would not be possible to deflect from AI to IE' over the curves shown in the figure; but if IA be assumed as the initial tangential direction, IE' would be the terminal direction, and the total deflection would be IA.IE', equal, by proposition IX., to IA.E'I in sign and magnitude.

If, then, a line **AE'** be drawn from A to E', and the angles A' and B' measured between **AE'** and the tangent-lines intersecting at I, the triangle AE'I would be similar to figure 7, problem V., and the deflection, AI.E'I = IA.IE', would be equal to (180°−(A + B') degrees to the left, or to that deflection, plus or minus a whole multiple of 360°.

The principle on which the method here explained is based is the same as that by which the notes of an alignment of curves (see problem IX.) are kept on railroad surveys.

The calculated change of direction or bearing, in passing from one tangential direction to another, is the algebraic sum of the deflections over the intervening arcs, and should always be checked by the observed change of bearing, taken with a magnetic needle.

When, as at the point C in the figure, the direction of the motion of translation is reversed, *without turning*, the cars on a railroad-track would move backwards, and the alignment would be that of a switch-back, sometimes used for temporary purposes, as, for instance, for a temporary track during the construction of a tunnel. When, as at D, there is a second switch-back, the two together form a "Y," and the cars would be turned forward again.

There are cases in mountainous countries where, by means

of two or three switch-backs, a great saving might be made in length of tunnels, grade, curvature, and bridging and trestling; and where auxiliary engines are necessary, switch-backs might be used to advantage. In keeping the field notes of such an alignment, it would be necessary, in order that the calculated and observed bearings should agree, either that the survey should be made backwards, or that the surveyor should allow for 180 degrees of curvature more than actually made on the track.

If the earth's surface were a plane, the deflection from the direction of an initial tangent to the direction of a terminal tangent, 100 or 1,000 miles distant, would be equal to the algebraic sum of all the deflections made between the tangents.

In the following examples not more than two curves will be considered, and it will be seen, unless the common tangent is a point, that there will always be a pair of common tangents and two pairs of curves, over which the required deflection may be made. The sum of the deflections will, however, be the same in all cases.

Article 2. For application of the values of I, in the following examples, to equations for compound and reversed curves, see tables of problems XXII. and XXVI.

Simple Curves.

Fig. 73, $I=+(A'+B')$. Fig. 74. $I=+(A'-B')$.
" 75, $I=-[180°-(A'+B')]$. " 76. $I=-[180°-(A'-B')]$.

Reversed Curves, Centres on Different Sides of **AB**, $I=(A'\pm B')$.

Fig. 102, $I=AI.IB=+Ia-Ib=+(A'+B')$.
" 103, $I=AI.IB=-Ia-Ib=-(A'+B')$.
" 104, $I=IA.IB=+Ia-Ib=+(A'-B')$.
" 105, $I=AI.BI=+Ia+Ib=+(A'-B')$.

Reversed Curves, Centres on the Same Side of **AB**, $I=180°\pm(A'\pm B')$.

Fig. 106, $I=AI.BI=+Ia+Ib=+[180°+(A'+B')]$.
" 107, $I=AI.BI=+Ia+Ib=+[180°-(A'+B')]$.
" 108, $I=AI.IB=-Ia-Ib=-[180°-(A'-B')]$.
" 109, $I=AI.IB=+Ia+Ib=+[180°+(A'-B')]$.

In figures 106 and 109, the deflection is to the right, and greater than 180 degrees; in figures 107 and 108, the deflections are between the same directions as in 106 and 109, but to the left, and less than 180 degrees, the deflections in one case being equivalent to those in the other, because their algebraic differences are equal to 360 degrees.

Compound Curves, Centres on the Same Side of AB, $I = (A' \pm B')$.

Fig. 83, $I = AI.IB = +Ia + Ib = +(A' + B')$.
" 114, $I = AI.IB = -Ia - Ib = -(A' + B')$.
" 112, $I = AI.IB = +Ia + Ib = +(A' + B')$.
" 86, $I = AI.IB = +Ia - Ib = +(A' + B')$.
" 80, $I = IA.IB = -Ia + Ib = +(A' - B')$.
" 81, $I = IA.IB = +Ia - Ib = +(A' - B')$.

Compound Curves, Centres on Different Sides of AB, $I = 180° - (A' \pm B')$.

Fig. 88¼, $I = IA.IB = +Ia - Ib = +[180° - (A' + B')]$.
" 88, $I = IA.IB = -Ia + Ib = +[180° - (A' + B')]$.
" 117, $I = AI.IB = +Ia + Ib = +[180° - (A' - B')]$.
" 118, $I = IA.BI = +Ia - Ib = -[180° - (A' - B')]$.

In all of the above examples, a plus sign prefixed to a deflection indicates that it is to the right, and a minus sign that it is to the left.

PROBLEM VI.

DEFLECTIONS OVER CROSSING CURVES.

In problem V., the connection between two tangents was supposed to be made over two arcs with a common intermediate tangent, and the deflection, from the initial to the terminal tangent, was shown to be equal to the algebraic sum, $Ia + Ib$, of the tangential deflections of the arcs.

In this problem, the deflection, from the initial to the terminal

tangential direction, is supposed to be made over two arcs intersecting at a point F; *from* the initial *to* the terminal tangent of the arc AF, thence *to* the initial tangential direction of the arc FB, and thence *to* the terminal tangential direction at the extremity of that arc.

The deflections over the crossing arcs AF and FB are represented here, and elsewhere throughout the book, by the symbols X*a* and X*b*, and the deflection at F, between the crossing tangents, by the letter F; I, as heretofore, being the angle of intersection between the tangents intersecting at I.

In the case of turn-outs from railroad-tracks for sidings, crossings, and switches, the acute value of F is called the frog-angle, because the crossing of the intersecting rails is made by means of a frog, and I is equal to the switch-angle, S, for single switches, and to twice S for double switches.

If, therefore, I is the deflection, from the initial tangent at A to the terminal tangent at B, over the arcs produced to contact with their common tangent, and (X*a*+F+X*b*) the same deflection over the crossing arcs, and between the tangents at their intersection, F,

$$(Xa+F+Xb)=I. \quad (1) \qquad (Xa+Xb)=(I-F). \quad (2)$$

It is required, in any particular figure or form of curves, to determine, by inspection, the value of I *in terms of* A′ *and* B′ *and the signs of* X*a*, X*b*, *and* F, *so that their algebraic sum may be equal to* I.

A being the initial, and B the terminal, point of the arcs, the acute angles A′ and B′ are to be measured between AB and the tangent lines intersecting at I, and the angle I will be equal to (A′ ± B′), or to 180°−(A′ ± B′), as explained in problem V.

It is shown in proposition XVII., article 5, that when, as in the case under consideration, the curves intersect, they will always be compound with reference to their common tangent, and therefore, by the rules of problem V.,

$$I=\begin{cases}(A'\pm B') \text{ when the centres are on the } \textit{same} \text{ side of } \mathbf{AB}.\\ 180°-(A'\pm B') \text{ when the centres are on different sides of } \mathbf{AB}.\end{cases}$$

The curves will always intersect in two points, and the angles, F, at the two crossings will be equal in value, but

contrary in sign. The values of Xa and Xb will be also different for the two crossings, but the algebraic sums of the three angles Xa, F, and Xb will, in either case, be equal, or equivalent, to I.

Representing the acute value of F by F′, and the obtuse value by $180°-F'$, F must have the value which will make the algebraic sum, $Xa+Xb+F$, equal to I.

To determine the equation for any particular figure, suppose, as in the last problem, the curves to be the centre lines of railroad-tracks, and the angles Xa, F, and Xb to be measured by the rotation of a car from A to B, the deflection, F, being made by means of a turn-table.

In figure 55¼, for instance, $I = AI.IB = (A' + B')$.

If a car headed in the direction AI should turn, over the tracks and turn-table, to the terminal direction IB, the deflections would be, $(+Xa$ right), $-(180°-F')$ *left*, and $(+Xb$ right), and the equation would be as follows:

$$I = AI.IB = +Xa-(180°-F')+Xb=(A'+B'). \quad (a)$$

By turning F′ degrees to the *right* at F, the car would move backwards from F to B, and on arrival at B would head in the direction BI opposite to IB, and the equation would therefore be:

$$I = AI.BI = +Xa+F'+Xb=+180°+(A'+B'). \quad (b)$$

To turn from AI to IB by the second crossing, the deflections would be, Xa right, $180°-F'$ right, and Xb right, and therefore,

$$I = AI.IB = +Xa+180°-F'+Xb=+360°+(A'+B'). \quad (c)$$

The deflection in this, as in (a), is to the right from AI to IB, and the two deflections, differing by 360 degrees, are equivalent. These examples suffice to explain the method by which the equation may be determined by inspection.

For application of the equations to formulæ for intersecting curves, sidings, crossings, turn-outs from double and single switches, and compound and reversed curves, see problems XVI. to XVIII., table III.

In the following examples, the right is positive, and the left negative:

Examples.

Fig. 51, $I = AI.IB = -Xa + F + Xb = +(A' - B')$.
" 51¼, $I = AI.IB = +Xa + F - Xb = +(A' - B')$.
" 51½, $I = AI.IB = +Xa - F + Xb = +(A' - B')$.
" 51¾, $I = AI.IB = +Xa - F + Xb = +(A' - B')$.
" 53, $I = AI.IB = -Xa - F - Xb = -[180° - (A' - B')]$.
" 54, $I = IA.BI = +Xa - F - Xb = -[180° - (A' - B')]$.
" 55, $I = AI.IB = +Xa - F + Xb = +(A' + B')$.
" 55¼, $I = AI.IB = +Xa - F + Xb = +(A' + B')$.
" 55½, $I = AI.IB = -Xa + F - Xb = +(A' - B')$.
" 56, $I = AI.IB = +Xa + F + Xb = +(A' + B')$.
" 57¼, $I = AI.BI = +Xa - F - Xb = -[180° - (A' + B')]$.
" 57½, $I = AI.BI = -Xa + F + Xb = -[180° - (A' + B')]$.
" 58, $I = AI.BI = -Xa + F + Xb = +[180° - (A' + B')]$.
" 59, I=Switch-angle=S. F=Frog-angle.
" 59¼, $S = +Xa + F - Xb$. $Xb - Xa = (F - S)$.
" 59, $S = -Xa + F - Xb$. $Xb + Xa = (F - S)$.
" 50, Siding, I=0, F=Frog-angle.
" 50, $1 = 0 = +Xa + F - Xb$. $Xb - Xa = +F$.
" 50½, $I = 0 = -Xa + F - Xb$. $Xa + Xb = +F$.
" 70, Crossing, Fa the greater frog-angle, Fb the lesser. $A' = 90°$. $B' = 90° - Fb$. $I = AI.IB = -[180° - (A' + B')] = -Fb$. $\therefore I = -Fb = -Xa - Fa + Xb$. $\therefore Xb - Xa = Fa - Fb$.

PROBLEM VII.

DEFLECTIONS BETWEEN CHORDS OF ARCS WITH CONNECTING TANGENTS.

(Figures 39 to 46.) AA′ and BB′ are arcs connected by a common tangent **A′B**; Ia and Ib are the tangential deflections over the arcs; $I = Ia + Ib$ is the tangential deflection from the initial tangent, through A, to the terminal tangent through B, and K is the deflection from the chord of the arc AA′ to the chord of the arc BB′.

Given two of the three deflections, I*a*, I*b*, *and* K; *required the third deflection.*

$$K = \frac{Ia + Ib}{2}. \quad (1) \qquad \frac{Ib}{2} = K - \frac{Ia}{2}. \quad (2)$$

In these equations, the signs of I*a* and I*b* are to be determined by the rules of problem V.; the sign of K is positive or negative, as the deflection is right or left, and the sum and difference is algebraic.

Examples.

For application of (2) to fitting an alignment of curves to the deflection points of a straight line survey, see problem IX., article 4.

In the following examples, the right is positive, and it will be observed that the formulæ apply whether the common tangent is a line or point:

Fig. 39, $K = AB.BB' = \frac{1}{2} AI.IB' = +\frac{1}{2} Ia + \frac{1}{2} Ib = +\frac{1}{2}(A' + B')$.
" 40, $K = AB.B'B = \frac{1}{2} AI.IB' = -\frac{1}{2} Ia + \frac{1}{2} Ib = -\frac{1}{2}(A' + B')$.
" 41, $K = AA'.BB' = \frac{1}{2} AI.IB' = +\frac{1}{2} Ia + \frac{1}{2} Ib = +\frac{1}{2}(A' + B')$.
" 42, $K = AB.BB' = \frac{1}{2} IA.IB' = +\frac{1}{2} Ia - \frac{1}{2} Ib = +\frac{1}{2}(A' - B')$.
" 43, $K = AA'.BB' = \frac{1}{2} IA.IB' = +\frac{1}{2} Ia - \frac{1}{2} Ib = +\frac{1}{2}(A' - B')$.
" 44, $K = AB.BB' = -\frac{1}{2} Ia + \frac{1}{2} Ib = 0$.
" 45, $K = AB.BB' = +\frac{1}{2} Ia - \frac{1}{2} Ib = 0$.
" 46, $K = AA'.B'B = \frac{1}{2} AI.B'I = +\frac{1}{2} Ia + \frac{1}{2} Ib = +90° + \frac{1}{2}(A' + B')$.

When the chords are on the same line, as in figures 44 and 45, or on parallel lines, when *t* is not a point, K is equal to zero, and the tangential deflections, I*a* and I*b*, must be equal, and with contrary signs, in order that the algebraic sum, I*a*+I*b*, may be equal to 0, and therefore the tangents through A and B must be parallel. It is to be observed, when I is equal to $180° \pm (A' \pm B')$, that K is equal to $90° \pm \frac{1}{2}(A' \pm B')$.

Demonstration.—It is assumed, in the case of a single arc, that the deflection from the tangent to the chord is half the tangential angle of the arc.

Prop. IV. (1). Fig. 41, AI.AA'+AA'.BB'+BB'.IB'=AI.IB'.
$(+\frac{1}{2}Ia)+ (+K) + (+\frac{1}{2}Ib)= I.$
Prob. V. (1). $Ia+Ib=I.$
Therefore, $K=-\frac{1}{2}Ia-\frac{1}{2}Ib+I=\frac{1}{2}I.$

PROBLEM VIII.

DEFLECTIONS BETWEEN CHORDS OF INTERSECTING ARCS.

Two curves from tangent-points, A and B, intersect at F; I is the deflection over the curves produced to contact with their common tangent; $(Xa+F+Xb)=I$, the same deflection over the crossing arcs, and between the crossing tangents, and Kf the deflection between the chords (**AF**) and (**FB**) of the crossing arcs.

Of the three deflections, I, F, *and* K*f, given two, to determine the third.*

Prob. VI. (1). $(Xa+Xb)=(I-F).$ (1) $Kf=\frac{1}{2}(F+I).$ (2)

Demonstration.—Problem IV. (1). The deflection, K*f*, from **AF** to **FB**, is equal to the algebraic difference, $\frac{1}{2}Xb-(-\frac{1}{2}Xa)$, of the bearings to the chords from their respective tangents, plus the deflection, F, from one tangent to the other, and therefore $Kf=\frac{1}{2}(Xa+Xb)+F$.

By problem VI., $\frac{1}{2}(Xa+Xb)=\frac{1}{2}(I-F)$, and therefore $Kf=\frac{1}{2}(F+I)$.

This sum is algebraic, and each pair of curves will cross at two points, for which the signs of F will be different, but the signs of F in (1) and (2) will always be different.

Example 1 (figure 50). This figure represents a siding to a railroad-track, for which $I=0$, and therefore $Kf=\frac{1}{2}F$.

Example 2 (figure 70). This figure represents a crossing from one railroad-track to another, for which I is equal to the greater frog-angle, F*a*, and F to the lesser frog-angle, F*b*, and therefore $Kf=\frac{1}{2}(Fa+Fb)$.

Example 3 (figure 59). This figure represents a turn-out from

a railroad-track, for which F is equal to the frog-angle; I to the switch-angle, S, and therefore $Kf = \frac{1}{2}(F + S)$.

Example 4 (figure 68). In this figure, which represents a turn-out from a double or three-throw switch, F″ is the middle frog-angle; I is equal to twice the switch-angle, S; and, as $+Xa - F'' + Xb = -I = -2S$, $(Xa + Xb) = (F'' - 2S)$, and $Kf = \frac{1}{2}(F'' + 2S)$.

For further examples, see problem XVI. with relation to intersecting curves, and problem XVIII. with relation to sidings, crossings, and turn-outs from single and double switches.

PROBLEM IX.

TO SURVEY, BY CHORD INTERSECTIONS, AN ALIGNMENT OF CURVES AND TANGENTS FOR A RAILROAD-TRACK.

1. *Definitions and Formulæ.—The degree of curvature* is the tangential deflection over an arc of which the chord is 100 feet, and is equal to the central angle between the radii through the extremities of the chord.

Stations.—The extremities of the 100′ chords are called stations, and marked by stakes numbered consecutively 0, 1, 2, 3, etc., each number representing a distance of 100 feet, measured over the 100′ chords.

The tangential deflection of an arc is the deflection from the initial to the terminal tangent, and *the chord deflection* is the deflection from the initial tangent to the chord through the extremities of the arc.

Let D be the degree, and r the radius of curvature of an arc; I, the tangential deflection, or angle of intersection; K, the chord deflection, and s the number of stations.

The relations between these quantities are expressed by the following formulæ:

$$\text{Sin. } \tfrac{1}{2} D = 50 \div r. \quad (1)$$

$$I = s.D. \quad (2)$$

$$K = s.\tfrac{1}{2} D. \quad (3)$$

$$i = 100 . \text{ sin. } \tfrac{1}{2} I \div \text{sin. } \tfrac{1}{2} D = 2r . \text{sin. } \tfrac{1}{2} I. \quad (4)$$

2. *Survey by* 100′ *chords* (figure 12). The instrument is placed at the initial point of the arc, marked 0 in the figure, and aligned on the tangent through that station. (3) The deflections, K, from the initial tangent to the chords from 0, to 1, 2, 3, . . . *s*, etc., are $1\times\frac{1}{2}$ D, $2\times\frac{1}{2}$ D, $3\times\frac{1}{2}$ D, . . . $s\times\frac{1}{2}$ D, etc.

A chain 100′ in length is used, and a rod, in the hands of the fore chainman, is aligned on the chord through I by the transitman, while the back chainman holds it at a distance of 100 feet from station 0 by means of the chain.

The rod is then aligned on the chord through station 2, and at a distance of 100 feet from station 1, and the same process is repeated to the terminal point of the arc, or until it is necessary to move the instrument.

If, when a deflection, $s\times\frac{1}{2}$ D, has been made to the chord through station *s*, it is desired to move the instrument, it is to be placed at station *s*, aligned on the chord through 0, and a deflection, $s.\frac{1}{2}$ D, made to the tangent at *s*.

The survey may then be made from the new tangent, by the process above described, until it is necessary again to remove the instrument. If the vernier reading is zero while on the initial tangential direction, and remains unchanged while moving the instrument, the reading on any other tangent will be the algebraic sum of the intervening tangential deflections.

It will, however, be found more convenient to have the reading zero at each tangent-point to which the instrument is moved, and this may be effected in the following manner.

Suppose the reading of the tangent at any station, as 0, to be zero, and the chord deflection, for the chord through stations 0 and *s*, to be $s\times\frac{1}{2}$ D. Move the instrument to station *s;* change the reading to $s\times\frac{1}{2}$ D on the *opposite side* of the zero of the graduated plate; align on the chord, and deflect until the reading is zero.

The instrument will then be aligned on the tangent at station *s*.

As a check on the work, and for convenience in platting, whether by scale or by latitudes and departures, it is necessary to calculate the changes of the bearings of the tangents, and to check the calculated bearings by comparison with the observed magnetic bearings. There is a constant liability to error by mistaking right for left; by the slipping of the plates, and

from other causes, and no notes can be considered reliable unless this precaution has been taken.

The method of calculating the bearings is shown in the following form of field notes:

R. R.... Line.... 188....

1 Sta.	2 Chord	3 Curve.	4 Deflections. Chord	4 Deflections. Tangent.	5 Tangential Bearings. Calculated.	5 Tangential Bearings. Observed.	6 Remarks.
0.50		B. C. 10° R.			−175° 00′	S. 5° 00′ W.	Commenced ....188..
1	50.05		2° 30′				
2	100.		7° 30′				
3	100.		12° 30′				
(.) .50	50.05	E. C. 10° R. B. T.	15° 00′	+30° 00′	−145° 00′	S. 35¼° W.	
4	50.						
(.) 5	100	E. T. B. C. 4° L.			−145° 00′	S. 36¾° W.	1° 00′ Diff. front and back sight.
6	100		2° 00′				
7	100		4° 00′				
8 (.)	100	E. C. 4° L. P. C. C. 8° L.	6° 00′ 6° 00′	−12° 00′	−157° 00′	S. 23¾° W.	
9	100		4° 00′				
10	100		8° 00′				
11 (.)	100	E. C. 8° L. B. T.	12° 00′ 12° 00′	−24° 00′	−181° 00′ +179° 00′	S. 1° 00′ E.	
12	100						From 12 to 16, 166 feet. Line 234′ short.
12.90	90′						
15.24	0						
16.00	76′				+179° 00′	S. 7¼° E.	Stopped ....188..
(.)		E. T. B. C. 8° R.					

Explanation of Form of Field Notes.—The symbols, or abbreviations, used in curvilinear surveys are as follows: B.C. and E.C. represent the beginning and end of a curve; B.T.

and E.T., the beginning and end of a tangent; P.T or T.P., a tangent-point, not on the extremity of an arc or tangent; P.C.C or P.R.C., a point of compound or reversed curvature; and R and L, right and left, with reference to deflections, the right being generally, as in the example above, the *positive* direction.

The survey is supposed to commence at station 0.50, with a 10-degree curve to the right. The magnetic bearing of the tangent at 0.50 is S. 5° 00′ W., or N. 175° 00′ W., or −175° 00′, counting from north 180 degrees to the right or left.

The calculated bearing should be expressed by this method rather than by quadrant, and may be readily compared with the observed bearing by subtracting from 180 degrees, when greater than 90 degrees. In column 4 are given the deflections from the tangent to the chords, calculated by (3). The deflection from the tangent at 0.50 to the chord through 0 and 3.50 is (3.50−0.50) multiplied by half the degree of curvature, or 15° 00′.

The instrument is moved to 3.50, aligned on the chord, and a deflection of 15 degrees made from the chord to the tangent.

The tangential deflection from 0.50 to 3.50 is double the chord deflection of 15 degrees, and to the right, and therefore equal to +30° 00′. The bearing of the tangent at 3.50 is the algebraic sum, (−175°) + (+30°) = (−145°), of the bearing at the preceding tangent-point, and the deflection over the arc.

This bearing is equivalent to S. 35° W., and the observed bearing is S. 35¼° W. In the same manner, the bearing of the terminal tangent of any arc is equal to the algebraic sum of the bearing of the initial tangent, and of the tangential deflection *from* the initial *to* the terminal tangent.

Station 3.50 is the beginning of a tangent which ends at 5.00. The calculated bearing is, of course, the same at both extremities of the tangent, but the observed bearings differ by one degree on account of local attraction. When there is a considerable amount of local attraction, the bearing should be taken at both extremities of the chord or tangent when the instrument is moved. In the same manner the chord deflections, and the tangential deflections and bearings, are determined for the remaining portions of the line.

It will be seen that the calculated bearing of the tangent

at station 11 is $-181°$, or N. 181° W., which is equivalent to $+179°$, or N. 179° E., or S. 1° 00′ E., which is also the observed bearing.

At station 12.90 a connection is made with station 15.24 of a line previously run, and $(12.90 = 15.24)$ is called the *equation* of the connection. As it is generally necessary to revise portions of every located line, and the equations of the connections are a constant source of error, the distance from the preceding *even* station to the point of connection, and thence to the *even* station next succeeding, should be always given, and also (in the column of remarks), the distance between the contiguous *even* stations on the old and revised lines.

In the example, 12.90 is supposed to be on the revised line, and the order of stations is as follows:

On the revised line, 11, 12, 12.90.
On the old line, 15.24, 16, etc.

The distance from 12 to $(12.90 = 15.24)$ is 90 feet, and the distance thence to station 16, 76 feet, so that the distance from 12 to 16 is 166 feet, or 234 feet *less* than is indicated by the numbers on the stakes.

If 12.90 were on the old line, the order of the stations would be as follows:

On the *revised* line, 13, 14, 15, 15.24.
On the old line, 12.90, 13, 14, etc.

In this case there would be two stations, 13, 14, and 15; the distances would be 24 feet from 15 to $(15.24 = 12.90)$, and 10 feet thence to 13, so that the distance would be 24 feet from 15 to 13, and 234 feet from 13 to 13, which is therefore 234 feet *greater* than indicated by the number of the stations. These connections should, if possible, be made on the tangents, and the notes should be made on the ground and given to the leveller for allowance in the grades.

3. *Survey by Long Chords.*—When it is required to mark only the extremities of the arcs, the lengths of the chords are to be calculated by (4), article 1, and the instrument moved from one extremity of each chord to the other, as in the following example, which is the same, as regards the alignment, with the example in article 2:

Sta.	Chord.	Curve.	Deflections.		Tangential Bearings.		Remarks.
			Chord.	Tangent.	Calculated.	Observed.	
0.50		B. C. 10° R.			−175° 00′	S. 5° 00′ W.	Commenced.....188....
3.50	297.0	E. C. 10° R.	15° 00′	+30° 00′	−145° 00′	S. 35¼° W.	
(.)			15° 00′				
5	150.0	E. T.					
(.)		B. C. 4° L.		0° 00′	−145° 00′	S. 36¼° W.	1° 00′ diff. front and back sight.
8	299.5	E. C. 4° L.	6° 00′				
(.)		P. C. C. 8° L.	6° 00′	− 12° 00′	−157° 00′	S. 23½° W.	
11	298.1	E. C. 8° L.	12° 00′		−181° 00′		
(.)					+179° 00′	S. 1° 00′ E.	
12.90		B. T.	12° 00′	− 24° 00′			From 12 to 16, 166′.
15.24	190′	T. P.					Line short, 234′.
(.)							
16	76′	E. T.					
(.)		B. C. 8° R.			+179° 00′	S. 14° E.	Stopped.....188....

In this example, the length of the chord, and the bearing of the tangent, are placed opposite the terminal point of each arc.

The bearing of the chord may be obtained by adding the chord, instead of the tangent, deflection, to the bearing of the initial tangent, and with these bearings and distances the

chords may be platted by scale, or by means of latitudes and departures, as in example 4, problem XI.

4. (figure 13).—*Given the lengths and deflections of any series of intersecting lines, considered as chords, and the deflection to the first from a given tangential direction; required the degrees of curvature, and the tangential deflections, of an alignment of curves with the intersections of the lines as points of compound or reversed curvature.*

In the example of article 3, the degrees and tangential deflections of the arcs are given, and the chords are calculated by (4).

In this example, the chords, and the deflections from one chord to the other, and the first chord deflection, are given, to determine the degrees of curvature and the tangential deflections of the arcs. The solution is based upon the following formulæ.

$\frac{1}{2}$ Ia and $\frac{1}{2}$ Ib are the chord deflections of any two contiguous arcs, with their common tangent-point at the intersection of their chords i_a and i_b. Da and Db are the degrees of curvature, and K is the deflection from one chord to the other.

The chord deflection $\frac{1}{2}$ Ia, and i_a, i_b, and K, are given, and the chord deflection $\frac{1}{2}$ Ib, and the degrees, Da and Db, and the numbers, s_a and s_b, of stations in the arcs, are determined by the following formulæ:

Problem VII. (2). $\frac{1}{2}$ Ib = K − $\frac{1}{2}$ Ia. (5)

Problem IX. (4). Sin. $\frac{1}{2}$ D = $\frac{100 \text{ . sin. } \frac{1}{2} \text{ I}}{i}$. (6)

Problem IX. (2). $s = \frac{\frac{1}{2} \text{ I}}{\frac{1}{2} \text{ D}}$. (7)

In (5), K and $\frac{1}{2}$ Ia are to be considered positive when to the right, and negative when to the left, and the algebraic difference, (K − $\frac{1}{2}$ Ia), will give the sign and value of $\frac{1}{2}$ Ib.

The application of these formulæ will be understood from the following example, referring to figure 13:

Sta. on Chord.	Chord i.	Deflection K.	Chord Deflection, $\frac{1}{2}I_2 = K_2 - \frac{1}{2}I_1$.	$\frac{1}{2}$ Degree of Curve, Sin. $\frac{1}{2}D = \frac{100 \cdot \text{Sin. } \frac{1}{2}I}{i}$.	S. $\frac{1}{2}I$. $\frac{1}{2}D$	Sta. on Curve.	Alignment.
0	1360	+25° 00′	+25° 00′	+1° 47′ (R.)	14.04	0.	B. C. 3° 34′ R.
13.60	920	+9° 00′	−16° 00′	−1° 43′ (L.)	9.30	14.04	B. R. C. 3°26′ L.
22.80	1640	−65° 00′	−49° 00′	−2° 38′ (L.)	18.58	23.34	B. C. C. 5° 16′ L.
39.20	1000	−22° 00′	+27° 00′	+2° 36′ (R.)	10.38	41.92	B. R. C. 5°12′ R.
49.20		+27° 00′	0° 00′			52.30	B. T.

Deflection from chord ending at 49.20 to tangent, 27° 00′ R.

In this example, the deflection, +25°, from the assumed tangential direction, at station 0, to the first chord, is the chord deflection of the first arc. The chord deflection of the second arc, from station 13.60, is the algebraic difference, (+9°) − (+25°) = (−16°), obtained by subtracting the chord deflection, (+25°), of the first arc, from the deflection, K, *from* the first *to* the second chord, and, in like manner, the chord deflection, ½I, of any arc is obtained by subtracting the chord deflection of the preceding arc from the deflection, K, between the chords of the two arcs.

At station 49.20, the chord deflection is zero, and the chord *from* that station coincides with the tangent.

From the chord deflections, ½I, thus calculated, the degrees of curvature, and the number of stations in the arcs, are determined by (6) and (7). In calculating the stations, it will be convenient to reduce degrees and minutes to degrees and decimals, by the rules of problem I. The number of stations in each arc having been determined, the numbers of the stations on the alignment may then be calculated, as shown in column 7 of the example. It will be seen that the distance measured over the 100′ chords is 110 feet greater than the distance over the long chords.

The alignment is given in the last column. When two con-

tiguous curves are reversed, another pair, with an intervening tangent, may be substituted by the rules of problem XXIV.

5. *Grade line surveys* are made to determine the practicability of a line from a summit, with a given maximum degree of curvature and rate of grade. Generally, the levels and rates of grade are exact, but the alignment is impracticable, and, consequently, no conclusions can be drawn as to the character or practicability of the location.

If, in such a survey, an initial direction of tangent is assumed, and the lines considered as the chords of an alignment of curves, compounding or reversing at the intersection points of the lines, it will serve as a useful check on the rate of curvature.

The chord deflections, as in the example of the preceding article, may be calculated by simple addition and subtraction, and the degrees of curvature, with sufficient exactness, by the following formula:

$$\tfrac{1}{2}\,D = \frac{100 \cdot \tfrac{1}{2}\,I}{i}. \quad (8)$$

The degree of curvature, thus calculated, should not exceed the maximum degree, and, when it is necessary, the chord should be extended so that D may not exceed that maximum.

When the signs of the chord deflections of two contiguous arcs are reversed, allowance should be made for the introduction of an intervening tangent by reducing the value of I in (8).

The calculations may be easily made, in the field, by the transit-man.

PROBLEM X.

RIGHT-ANGLED TRIANGLES—DEPARTURES AND LATITUDES.

(*For demonstration see proposition XIII.*)

(Figure 18.) A is the initial, and B the terminal, point of a line (**AB**).

AN and **AW** are co-ordinate axes through the initial point, A; the direction of the axes being such that the deflection, AW.AN, is $+90°$.

a is the length of the line (**AB**), and (AN.AB = A) is the

bearing from the axis AN, estimated, right and left, from 0° to 180°, and not by quadrant.

The departure, d, and the latitude, l, are the perpendicular distances, right or left, to the terminal point, B, the one from axis **AN**, and the other from the axis **AW**.

These distances are positive when measured in one direction, as to the right, and negative when measured in the opposite direction.

(I.) *Given a and* A, *to determine d and l;*
(II.) *Given d and l, to determine a and* A.

The solution is based on the following proposition:

(**AB**) is the hypothenuse of a right-angled triangle, of which d is the perpendicular, and l the base.

The bearing A is equal to the acute angle of this triangle opposite d, when l is positive, and to 180 degrees minus that angle when l is negative.

The sign of A is the *same* as the sign of d.

To determine d and l when a and A *are known.*

$$d = a.\sin.\ A. \qquad l = a.\cos.\ A. \quad (1)$$

In these equations, d will have the same sign as A, and l will be positive when A is less than 90 degrees, and negative when it is greater.

To determine a and A *when d and l are known.*

a and A may be determined by (2) or by (3), as follows:

$$\left.\begin{array}{l} \text{Tang. } A = d \div l. \\ a = d \div \sin.\ A = l \div \cos.\ A. \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l} a = \sqrt{d^2 + l^2}. \\ \text{Sin. } A = d \div a. \quad \text{Cos. } A = l \div a. \end{array}\right\} \quad (3)$$

In these equations, A will be equal to the acute angle corresponding to the tangent, sine, or cosine, when l is positive, and to 180 degrees minus the acute angle when l is negative, and the sign of A will be the same as the sign of d.

In the case of a single right-angled triangle, it is, as a rule, necessary to consider only the magnitudes of the quantities; but to determine the position of B, or the distance and direction from another point, A, the magnitudes and signs must be considered.

Example 1. AN.AB = (−150°); (**AB**)=1,000; required the departure and latitude.

$$(1)\quad \begin{cases} d = 1{,}000 . \sin.\ (-150°) = 1{,}000 \times .500 = (-500). \\ l = 1{,}000 . \cos.\ (-150°) - 1{,}000 \times .866 = (-866). \end{cases}$$

The departure, *d*, is negative because the bearing is negative, and the latitude is negative because the bearing is greater than 90 degrees. Considering AN to be north and AW west, and the right positive, the terminal point, B, would be 500 left, or west, of **AN**, and 866 left, or south, of **AW**.

Should the bearing be (−30°), instead of (−150°), the sign and magnitude of the departure would be the same, but the sign of *l* would be positive, and B would be 866 right, or north, of **AW**.

Example 2. The departure, *d*, is (−500), and the latitude (−866).

Tang. (A = −150°) = $d \div l$ = .500 ÷ .866 = .5773.

$a = d \div \sin.\ A = l \div \cos.\ A$ = 500 ÷ .500=866 ÷ .866=1,000.

If the right is positive, the bearing A is to the left of AN because the point B is to the left of **AN**, and is greater than 90 degrees because B is to left, or south, of **AW**.

PROBLEM XI.

DEPARTURES AND LATITUDES.

(*For demonstration see proposition XIII.*)

(**AB**), (**BC**), (**CD**), . . . (**WX**), (**XY**), and (**YZ**) are a series of lines extending from A to Z. The initial point, A, of the series is the *origin of co-ordinates*, and the line from A to any point is the *closing line* to that point.

The *co-ordinates* of any point, Z, are the perpendicular distances to Z, right or left, from the co-ordinate axes **AN** and **AW**.

The distance from **AN** is the *ordinate*, and the distance from **AW** the *abscissa*, and the ordinate is, therefore, equal to the departure, and the abscissa to the latitude, of the closing line, from A to Z.

and E.T., the beginning and end of a tangent; P.T or T.P., a tangent-point, not on the extremity of an arc or tangent; P.C.C or P.R.C., a point of compound or reversed curvature; and R and L, right and left, with reference to deflections, the right being generally, as in the example above, the *positive* direction.

The survey is supposed to commence at station 0.50, with a 10-degree curve to the right. The magnetic bearing of the tangent at 0.50 is S. 5° 00′ W., or N. 175° 00′ W., or −175° 00′, counting from north 180 degrees to the right or left.

The calculated bearing should be expressed by this method rather than by quadrant, and may be readily compared with the observed bearing by subtracting from 180 degrees, when greater than 90 degrees. In column 4 are given the deflections from the tangent to the chords, calculated by (3). The deflection from the tangent at 0.50 to the chord through 0 and 3.50 is (3.50−0.50) multiplied by half the degree of curvature, or 15° 00′.

The instrument is moved to 3.50, aligned on the chord, and a deflection of 15 degrees made from the chord to the tangent.

The tangential deflection from 0.50 to 3.50 is double the chord deflection of 15 degrees, and to the right, and therefore equal to +30° 00′. The bearing of the tangent at 3.50 is the algebraic sum, (−175°) + (+30°) = (−145°), of the bearing at the preceding tangent-point, and the deflection over the arc.

This bearing is equivalent to S. 35° W., and the observed bearing is S. 35¼° W. In the same manner, the bearing of the terminal tangent of any arc is equal to the algebraic sum of the bearing of the initial tangent, and of the tangential deflection *from* the initial *to* the terminal tangent.

Station 3.50 is the beginning of a tangent which ends at 5.00. The calculated bearing is, of course, the same at both extremities of the tangent, but the observed bearings differ by one degree on account of local attraction. When there is a considerable amount of local attraction, the bearing should be taken at both extremities of the chord or tangent when the instrument is moved. In the same manner the chord deflections, and the tangential deflections and bearings, are determined for the remaining portions of the line.

It will be seen that the calculated bearing of the tangent

at station 11 is $-181°$, or N. 181° W., which is equivalent to $+179°$, or N. 179° E., or S. 1° 00′ E., which is also the observed bearing.

At station 12.90 a connection is made with station 15.24 of a line previously run, and $(12.90 = 15.24)$ is called the *equation* of the connection. As it is generally necessary to revise portions of every located line, and the equations of the connections are a constant source of error, the distance from the preceding *even* station to the point of connection, and thence to the *even* station next succeeding, should be always given, and also (in the column of remarks), the distance between the contiguous *even* stations on the old and revised lines.

In the example, 12.90 is supposed to be on the revised line, and the order of stations is as follows:

On the revised line, 11, 12, 12.90.
On the old line, 15.24, 16, etc.

The distance from 12 to $(12.90 = 15.24)$ is 90 feet, and the distance thence to station 16, 76 feet, so that the distance from 12 to 16 is 166 feet, or 234 feet *less* than is indicated by the numbers on the stakes.

If 12.90 were on the old line, the order of the stations would be as follows:

On the *revised* line, 13, 14, 15, 15.24.
On the old line, 12.90, 13, 14, etc.

In this case there would be two stations, 13, 14, and 15; the distances would be 24 feet from 15 to $(15.24 = 12.90)$, and 10 feet thence to 13, so that the distance would be 24 feet from 15 to 13, and 234 feet from 13 to 13, which is therefore 234 feet *greater* than indicated by the number of the stations. These connections should, if possible, be made on the tangents, and the notes should be made on the ground and given to the leveller for allowance in the grades.

3. *Survey by Long Chords.*—When it is required to mark only the extremities of the arcs, the lengths of the chords are to be calculated by (4), article 1, and the instrument moved from one extremity of each chord to the other, as in the following example, which is the same, as regards the alignment, with the example in article 2:

determine the co-ordinates of the points of intersection, and the bearing and distance from 0 to 12.50, and from 12 to 1.

Station.	Bearing.		Distance.	Departure.	Ordinate.	Latitude.	Abscissa.
0	+10° 00′	N. 10° E.	100	+17.4	0.0	+98.5	0.0
1	+50° 00′	N. 50° E.	400	+306.4	+17.4	+257.1	+98.5
5	+150° 00′	S. 30° E.	500	+250.0	+323.8	−433.0	+355.6
10.	+180° 00′	South.	200	0.0	+573.8	−200.0	−77.4
12.	−170° 00′	S. 10° W.	50	−8.2	+573.8	−49.3	−277.4
12.50				+565.6	+565.6	−326.7	−326.7

In this table, opposite to the initial station of each line, is placed the bearing, length, departure, and latitude of the line and the co-ordinates of the station.

The co-ordinates of the origin (station 0), are zero; the ordinate of station 1 is the departure, and the abscissa the latitude, of the line of which station 1 is the terminal point; the ordinate or abscissa of any other station is equal to the algebraic sum of the ordinate or abscissa, and of the departure or latitude, on the preceding line.

The bearing, Z, and the distance, z, of the closing line, from station 0 to station 12.50, are determined in the following manner:

The departure, $e_1 = (+565.6)$. The latitude, $n_1 = (-326.7)$.

Prob. X. (2) $\begin{cases} \text{Tang.}(Z = +120°) = (+565.6) \div (-326.7) = -1.731. \\ z = 565.6 \div \sin. 120° = 326.7 \div \cos. 120° = 653.2. \end{cases}$

There are two angles, 120° and 60°, corresponding to the numerical value of the tangent, and Z is the greater, because the latitude is negative, and has the same sign with the departure.

The line from station 0 to 12.50 is therefore 653.2 in length, and bears 120° to the right of north, or S. 60° E.

The same result would be obtained from (3), problem X.

To determine the bearing and distance from station 12 *to* 1.

(2) { The co-ordinates of station 1 are (+17.4) and (+98.5).
The co-ordinates of station 12 are (+573.8) and (−277.4).

(3) { The departure from 12 to 1 is (+17.4)−(+573.8)=(−556.4).
The latitude from 12 to 1 is (+98.5)−(−277.4)=(+375.9).

Station 1 is therefore 556.4 west, and 375.9 north, of station 12, and the bearing, Z, and length, z, are determined as follows:

Prob. X. (2) { Tang.(Z=−55° 57′)=(−556 4)÷(+375.9)=1.4802.
z = (556.4) ÷ sin. Z = (375.9) ÷ cos. Z = 671.5.

Z is the lesser of the two angles, 124° 03′ and 55° 57′, corresponding to the tangent 1.4802, because the latitude is positive, and is negative because the departure is negative.

4. *To determine the co-ordinates of the extremities of the chords in a given alignment of curves.*

For this purpose the bearing of the initial tangent must be given, and also the deflection from the tangent to the first chord; the deflection from one chord to another, and the lengths of the chords, must then be calculated by the following formulæ:

The *chord deflection* of an arc is the deflection from the tangent to the chord, and is equal to half the tangential deflection over the arc. If i is the length, ½I the chord deflection, D the degree, and r the radius of curvature,

Prob. IX. (4) (i = 100.sin. ½I ÷ sin. ½D = 2r.sin. ½I.
Sin. ½I = i.sin. ½D ÷ 100 = ½i ÷ r.) (a)

If ½I′ and ½I″ are the chord deflections of two arcs with a common tangent, and if K is the deflection from one chord to the other,

Prob. VII. (1). K = ½I′ + ½I″. (b)

A tangent is to be regarded as a chord for which the degree, D, and the angle, ½I, are zero, and the sum, ½I′ + ½I″, in (b), is algebraic. If, therefore, the chord deflections, ½I,

equal to half the tangential deflections of the arcs, are given, the deflection, K, from one chord to another, and the chord lengths, are determined by (a) and (b), and as the bearing of the first chord is equal to the algebraic sum of the bearing of the tangent and the first chord deflection, the bearing of each chord may be determined by (1), and the co-ordinates of the extremities of the chords by (3), of this problem.

Example 1 (figure 6). This figure represents two circles, of equal radii, reversing at E, and circumscribed on hexagons, the sides of which are equal to the radius:

Initial Station.	Chord Deflection	Deflection between Chords.	Bearing of Chord.	Length of Chord.	Departure of Chord.	Ordinate of s.	Latitude of Chord.	Abscissa of s.
Tangent.								
AN.			o					
A	+30°	+30°	+30°	1000	+500	o	+866	o
B	+30°	+60°	+90°	1000	+1000	+500	o	+866
C	+30°	+60°	+150°	1000	+500	+1500	−866	+866
D	+30°	+60°	+210° / −150°	1000	−500	+2000	−866	o
E	−30°	o°	−150°	1000	−500	+1500	−866	−866
F	−30°	−60°	−210° / +150°	1000	+500	+1000	−866	−1732
G		+150°			+1500	+1500	−2598	−2598

The co-ordinates are to be determined from the initial tangent, **AN**, as an axis, and from A as the origin of co-ordinates.

The chord deflections are, each, 30 degrees, and the deflections, K, from one chord to the other 60 degrees, except at E, where the signs of the chord deflections are different, and the algebraic sum is therefore zero.

The bearing of AB is equal to the chord deflection (+30°).

The bearing of any chord, as CD, is the algebraic sum, $(+90°) + (+60°) = (+150°)$, of the bearing of the preceding chord, BC, and the deflection, K, from that chord to CD.

The chords are equal to 1,000′, and the departures and latitudes are 1,000 times the sines and cosines of the bearings.

The signs of the departures are the same with the signs of the bearings, and the latitudes are positive when the bearings are less than 90 degrees, and negative when they are greater.

The ordinate of A is zero; the ordinate of B is the departure of (**AB**); the ordinate of C is the sum of the ordinate of B and the departure of (**BC**); and the co-ordinate of any point is the algebraic sum of the co-ordinate, and departure or latitude of the preceding point.

The ordinate of the terminal point, G, is the algebraic sum of the departures, and the abscissa of the latitudes, of the lines between A and G; and as these co-ordinates are equal to the departure and latitude of the closing line (**AG**), the bearing and length of that line may be determined from (2) or (3), problem X., in the following manner:

The co-ordinates of G are, $e_G = +1{,}500$, and $n_G = -2{,}598$.

$$(2)\quad \begin{cases} \text{Tang.}\,(G = +150°) = (e_G \div n_G) = (+1{,}500) \div (-2{,}598) = .5774. \\ g = e_G \div \sin.\,G = n_G \div \cos.\,G \qquad\qquad = 3{,}000. \end{cases}$$

The same result would be obtained from (3).

In (2), the numerical value, .5774, of the tangent corresponds to two angles, 30° and 150°, but G is the *greater* value, because n_G is negative, and G is therefore to the left, or south, of **AW**, and the bearing is +150°, or S. 30° E., the sign being fixed by the sign of e_G, which shows that G is to the right, or east, of **AN**.

(3) *To determine the length and bearing of the line* (**FC**), *extending from* F TO C (NOT *from* C TO F).

The departure of (**FC**) is the algebraic difference, $(e_C - e_F)$, of the ordinates of its extremities, and the latitude the algebraic difference, $(n_C - n_F)$, of the abscissas, the co-ordinates of the initial point, F, being always the subtractive terms of the algebraic differences. The length and bearing of the line may

therefore be determined by (2) or (3), problem X., in the following manner:

$$e_C = (+1{,}500). \qquad n_C = (+866).$$
$$e_F = (+1{,}000). \qquad n_F = (-1{,}732).$$
$$(e_C - e_F) = (+500). \qquad (n_C - n_F) = (+2{,}598).$$

(2) $\begin{cases} \text{Tang. (AN.FC} = +10^\circ\ 54') = (e - e_F) \div (n_C - n_F) = .1925. \\ (\mathbf{FC}) = 500 \div \text{sin. } 10^\circ\ 54' = 2{,}598 \div \text{cos. } 10^\circ\ 54' = 2{,}646. \end{cases}$

In (2), the tangent, .1925, corresponds to two angles, $+10^\circ\ 54'$ and $+169^\circ\ 06'$, and the lesser angle is the value of the required bearing, because the latitude, $(n_C - n_F) = (+2{,}598)$, is positive. The sign of the bearing is positive, because the sign of the departure is positive.

The departure and latitude of the line (CF), *from* C *to* F, would be negative, and therefore the bearing, AN.CF, would be $-169^\circ\ 06'$, or S. $10^\circ\ 54'$ W.

The co-ordinates of the point D are $(+2{,}000)$ and 0; and, therefore, D is 2,000 feet right, or east, of **AN**, and *on* the line **AW**, so that (**AD**) is equal to 2,000 feet, which is the diameter of the circle in which the hexagon is inscribed.

Example 2 (figure 15). *Given the degree,* D, *and the angle,* I, *of an arc; required the co-ordinates, or offsets, to the extremities of sub-chords of given length, from the chord of the arc as the axis* SN.

The bearing of the tangent is equal to $\frac{1}{2}$ I.

The deflection, $\frac{1}{2}$ I_1, *from* the tangent *to* the first sub-chord, i_1, is obtained by the following equation:

$$\text{Sin. } \tfrac{1}{2} I_1 = i_1 . \text{sin. } \tfrac{1}{2} D \div 100.$$

The deflection from one chord to another is equal to half the sum of the angles subtended by the chords, or to the sum of the chord deflections.

From the bearing of the first sub-chord, the deflections between the sub-chords, their lengths, and the co-ordinates of their extremities, may be determined, as in the following example:

Station.	Deflections between Chords (K).	Sub-Chords. Bearings. From Chord (2.8).	Sub-Chords. Bearings. By Quadrant.	Sub-Chords. Lengths.	Departures.	Ordinates (e).	Latitudes.	Abscissas (n).	
2	+4°	−24°	N. 24° W.	100	−34.20	0.00	+93.96	0.00	Tangent bearing at 2.
3	+8°	−20°	N. 20° W.	100	−20.79	−34.20	+97.82	+93.96	Origin at 2.
4	+8°	−12°	N. 12° W.	100	−6.98	−54.99	+99.75	+191.78	
5	+8°	−4°	N. 4° W.	100	+6.98	−61.97	+99.76	+291.53	
6	+8°	+4°	N. 4° E.	100	+20.79	−54.99	+97.82	+391.29	
7	+8°	+12°	N. 12° E.	100	+34.20	−34.20	+93.97	+489.11	
8	+8°	+20°	N. 20° E.	100	+46.95	0.00	+88.29	+583.08	
9	+8°	+28°	N. 28° E.	100	+58.78	+46.95	+80.90	+671.37	
10	+8°	+36°	N. 36° E.	100	+69.47	+105.73	+71.93	+752.27	
11	+6° 12′	+44°	N. 44° E.	100	+42.38	+175.20	+35.23	+824.20	
+55		+50° 12′	N. 50° 12′ E.	55		+217.48		+859.43	

In this example, the degree of curvature is 8 degrees, the angle is 48 degrees, the sub-chords are 100 feet, and the chord from which the offsets are to be measured passes through stations 2 and 8.

The bearing of the tangent from the chord (2, 8) is half the angle of the arc, or (−24°). The deflection from the tangent to the first sub-chord is half the degree of curvature, and the deflection from one chord to another is equal to the degree of curvature, *except* the last chord, which subtends an angle of 4° 24′, and, therefore, the deflection from the chord (10, 11) to the chord (11, 11.55), being half of the sum, (8°+4° 24′), of the angles subtended by the chords, is 6° 12′.

The bearing of the chord from any station is equal to the algebraic sum of the preceding bearing and the deflection, K, at the station; the departure and latitude of a course is found by multiplying the sine and cosine of the bearing by the length of the chord; the co-ordinates of station 2, which is the origin of co-ordinates, are equal to zero; the co-ordinates of station 3 are equal to the departure and latitude of the chord from 2 to 3, and the co-ordinate, or offset, to any station is equal to the offset to the preceding station added to the departure or latitude from that station. The chord from one station to another is the hypothenuse of a right-angled triangle, of which the perpendicular is the difference of the ordinates (or eastings) of the extremities, and the base is the difference of the abscissas (or northings).

The middle ordinate, or ordinate of the middle station, 5, is 61.97, and the length of the chord from 2 to 8 is the abscissa, (583.08), of station 8.

Example 8 (figure 16). *To determine the offsets, from the initial tangent, to an arc of given degree of curvature.*

The method is the same as in the preceding example, except that the tangent of the arc, instead of the chord, is assumed as the axis from which the bearings and ordinates are measured.

Station.	Chord Deflection (½I).	Deflection between Chords (K).	Bearing of Chords (B).	Length (*l*).	Departure (*d*).	Ordinate (*e*).	Latitude (*f*).	Abscissa (*n*).
Tangent.								
0	+4°	+4°	0	100	+6.98	0.00	+99.76	0.00
1	+4°	+8°	+4°	100	+20.79	+6.98	+97.81	+99.76
2	+4°	+8°	+12°	100	+34.20	+27.77	+93.97	+197.57
3	+4°	+8°	+20°	100	+46.95	+61.97	+88.30	+291.54
4	+4°	+8°	+28°	100	+58.78	+108.92	+80.90	+379.84
5	+4°	+8°	+36°	100	+69.47	+167.70	+71.93	+464.74
6		+4°	+44°		+237.17	+237.17	+532.67	+532.67
								8° Curve.

Example 4 (figure 17). *To lay off an alignment of curves and tangents by offsets from an axis at a given angle with the tangent at the initial station, and from an origin at a given distance from that station.*

In the example below, the offsets to the initial station, (0.75), from the origin, are −180′ and +200′, and the deflection from the assumed axis to the initial tangent through 0.75 is 18° 30′.

In the three first columns are given the stations, degrees, and tangential deflections of the different arcs, and from these data it is required to find the bearings and lengths of the chords, and the offsets to their extremities from the given axes of co-ordinates.

Initial Station of Arc (s_1).	Degree of Curvature (D_1).	Tangential Deflection (I_1).	Chord Deflection ($\frac{1}{2} I_1$).	Deflection between Chords $K_1 = \frac{1}{2} I_0 + \frac{1}{2} I_1$.	Bearing of Chord $B_1 = B_0 + K_1$.	Chord $c_1 = \frac{100 \cdot \sin. \frac{1}{2} I_1}{\sin. \frac{1}{2} D_1}$	Departure $d_1 = l_1 \cdot \sin. B_1$.	Ordinate of s_1 $e_1 = e_0 + d_0$.	Latitude $l_1 = c_1 \cdot \sin. B_1$.	Abscissa $n_1 = n_0 + l_0$.
Origin.			0° 0′	0° 0′	−13° 30′		−180.00	0.00	+200.00	0.00
0.75	B. C. 4° R.	+9° 00′	+4° 30′	+4° 30′	−9° 00′	224.81	−35.17	−180.00	+222.04	+200.00
3.00	B. C. C. 8° R.	+32° 58′	+16° 29′	+20° 59′	+11° 59′	406.67	+84.41	−215.17	+397.81	+422.04
7.12	B. T.	0° 00′	0° 00′	+16° 29′	+28° 28′	500.00	+238.27	−130.76	+439.58	+819.85
12.12	B. C. 10° L.	−40° 00′	−20° 00′	−20° 00′	+8° 28′	392.42	+57.73	+107.51	+388.15	+1259.43
16.12	B. R. C. 2° R.	+6° 00′	+3° 00′	−17° 00′	−8° 32′	299.88	−44.53	+165.24	+296.55	+1647.58
19.12	B. T.	0° 00′	0° 00′	+3° 00′	−5° 32′	300.00	−28.96	+120.71	+298.00	+1944.13
22.12	E. T.	+7° 58′	+3° 59′	+7° 59′	−13° 30′ +7° 58′		+91.75	+91.75	+242.13	+2242.13

The successive steps by which the calculation is made are as follows:

In column 4 place the chord deflections, equal to half the corresponding tangential deflections in column 3, and *with the same sign.*

In column 5 place the deflections, K, from one chord direction to another. The value of K, opposite the first chord deflection, is the same with that deflection. The value of K, opposite any other chord deflection, is equal to the algebraic sum of that and the preceding chord deflection.

In column 6 are the bearings of the chords. The bearing of the first chord is equal to the algebraic sum of the first value of K and of the bearing of the initial tangent when that tangent is not the axis, and equal to K when the bearing of the initial tangent is equal to zero. The bearing of any other chord direction is equal to the algebraic sum of the angle K, on the same horizontal line, and of the preceding bearing.

In column 7 are the lengths, *i*, of the chords, calculated by the formula, $i = 100.\ \sin.\ \frac{1}{2}\mathrm{I} \div \sin.\ \frac{1}{2}\mathrm{D}$, in which $\frac{1}{2}$I is the chord deflection, and D the degree of curvature.

The sum of the angles, K, should be equal to the bearing of the last chord, minus the bearing of the initial tangent when that bearing is not zero; the ordinate, (+91.78), of the terminal station, 22.12, is the sum of all the departures, and the abscissa, (+2242.13), of all the latitudes, and the ordinate or abscissa of any point, is the algebraic sum of the departure or latitude, and ordinate or abscissa, opposite to the preceding station.

The co-ordinates of the chords are generally calculated by means of semi-tangents, or distances from the initial point of the arc to the intersection of the tangents, and thence to the terminal point; the change of bearing, from one semi-tangent to the other, being the tangential deflection of the arc. This method, however, involves a somewhat greater amount of calculation than the above, and the intersection of the tangents is not upon the alignment.

The calculation of the co-ordinates of an alignment of railroad curves should be in sections of not more than one mile, so that in case of a change of a portion of the line, or an error of calculation, it may not be necessary to recalculate the co-ordinates of the whole line.

PROBLEM XII.

RIGHT AND OBLIQUE TRIANGLES.

(For demonstration see proposition X.)

In a triangle there are three sides and three angles.

Given a side and two other parts; required the remaining parts.

1 (figure 24). *Right-angled triangles* are triangles of which one angle is a right angle, equal to 90 degrees.

Let h be the hypothenuse, or side opposite the right angle; p the perpendicular, and P the angle opposite to p; b the base, and b the angle opposite to b.

The right angle and the two parts in column 1 of the table below being known, the parts in column 2 are determined by the formulæ in column 3.

1 Given.	2 Required.	3 Formulæ.	
P or B	B or P	$B = 90° - P$. $P = 90° - B$	(1)
P or B and h	p and b	$p = h \times \sin. P = h \times \cos. B$; $b = h \times \cos. P = h \times \sin. B$	(2)
P or B and b	p and h	$p = b \times \tan. P = b \times \cot. B$; $h = b \div \cos. P = b \div \sin. B$	(3)
P or B and p	b and h	$b = p \times \cot. P = p \times \tan. B$; $h = b \div \cos. P = b \div \sin. B$	(4)
h and b	B, P, and p	$\sin. B = \cos. P = b \div h$; $p = \sqrt{(h+b) \cdot (h-b)}$	(5)
h and p	B, P, and b	$\cos. B = \sin. P = p \div h$; $b = \sqrt{(h+p) \cdot (h-p)}$	(6)
b and p	B, P, and h	$\cot. B = \tan. P = p \div b$; $h = \sqrt{b^2 + p^2}$	(7)

2 (figure 27). *Oblique triangles* are triangles of which each angle is either greater or less than 90 degrees.

In the solution of the problem for oblique triangles, there will be four different cases, in which the given parts will be as follows:

1. A side and the two adjacent angles.
2. Two sides and an angle opposite to one of the sides.
3. Two sides and the included angle.
4. The three sides.

The parts in column 2 of the table below being known, the parts in column 3 are determined by the formulæ in column 4.

1 Case.	2 Given.	3 Required	4 Formulæ.	
	Two angles.	The third	$A+B+C=180°$.	(8)
1	A, c, B	a, C, b	$\left\{\begin{array}{l} C=180°-(A+B) \\ a=\sin. A \frac{c}{\sin. C}. \quad b=\sin. B \frac{c}{\sin. C}. \end{array}\right\}$	(9)
2	a, b, A	C, B, c	$\left\{\begin{array}{l} \sin. B=b\frac{\sin. A}{a}. \quad \sin. C=c\frac{\sin. A}{a}. \\ c=\sin. C \frac{a}{\sin. A}=\sin. C \frac{b}{\sin. B}. \end{array}\right\}$	(10)
3	a, C, b	A, c, B	$\left\{\begin{array}{l} \text{Tan. } Z=\cot. \frac{1}{2} C \frac{a-b}{a+b}.\ * \\ A=(90°-\frac{1}{2} C+Z). \quad B=(90°-\frac{1}{2} C-Z). \\ c=\sin. C \frac{a}{\sin. A}=\sin. C \frac{b}{\sin. B}. \end{array}\right\}$	(11)
4	a, b, c	C, B, A	$\left\{\begin{array}{l} x=\frac{(b+c) . (b-c)}{a}.\ † \\ \cos. C=\frac{\frac{1}{2}(a+x)}{b}. \quad \cos. B=\frac{\frac{1}{2}(a-x)}{c}.\ ‡ \\ \sin. A=a.\frac{\sin. C}{c}=a.\frac{\sin. B}{b}. \end{array}\right\}$	(12)

* a is the greater of the two sides, a and b.
† b is the greater of the two sides, b and c.
‡ If x is greater than a, B is greater than 90°.

3. *Right-angled triangles—Example* 1 (figure 24). The angle BIIP is 90°; the angle (HPB = P) is 30°; the base, b, is 200°; required the perpendicular, p, and the hypothenuse, h.

$$(3)\ \begin{cases} p = b.\text{tang. } P = 200 \times .517 = 115.4. \\ h = b \div \text{cos. } P = 200 \div .866 = 230.9. \end{cases}$$

Example 2. $p = 200$; $b = 400$; required B, P, and h.

(7) Cot. $(B = 63°\ 26') = \text{tan. } (P = 26°\ 34') = p \div b = 200 \div 400 = .50.$

$h = (p^2 + b^2)^{\frac{1}{2}} = (200{,}000)^{\frac{1}{2}} = 447.2.$

(2) $h = b \div \text{cos. } P = 400 \div .8944 = 447.2.$

Oblique triangles—Example 3, *Case* 1 (figure 25). Given $A = 30°$, $B = 40°$, and the included side, $c = 500$; required C, a, and b.

(9) $C = 180° - (A + B) = 110°.$

$a = \text{sin. } A.(c \div \text{sin. } C) = .50.(500 \div .9397) = 266.1.$

$b = \text{sin. } B.(c \div \text{sin. } C) = .6428.(500 \div .939) = 342.0.$

Example 4, *Case* 3. Given $a = 540$, $b = 450$, and the included angle, $C = 80$; required B, A, and c.

(11) Tan. $(Z = 6°\ 11') = \text{cot. } \frac{1}{2}C.(a+b) \div (a-b) = 1.192.\frac{90}{990} = .1084.$

$A = 90° - \frac{1}{2}C + Z = 50° + 6°\ 11' = 56°\ 11'.$

$B = 90° - \frac{1}{2}C - Z = 50° - 6°\ 11' = 43°\ 49'.$

$b = \text{sin. } C.\ (a \div \text{sin. } A) = .985.(540 \div .8308) = 640.1.$

In this example, a is the greater of the sides a and b, and A the greater of the angles A and B.

Example 5, *Case* 4. $a = 6$, $b = 5$, $c = 4$; required A, B, and C.

(12) $x = (b + c).(b - c) \div a = 9 \div 6 = 1.50.$

$\text{Cos. } (C = 41°\ 25') = \frac{1}{2}(a + x) \div b = 3.75 \div 5 = 0.75.$

$\text{Cos. } (B = 55°\ 46') = \frac{1}{2}(a - x) \div c = 2.25 \div 4 = .5625.$

$A = 82°\ 49' = 180° - (B + C).$

In order that $(b - c)$ may be positive, b must be the greater of the sides b and c; and if x is greater than a, $(a - x)$ is negative, and also cos. B, and the angle B is obtuse.

PROBLEM XIII.

FORMULÆ FOR ANGLES AND LINES OF CIRCULAR ARCS.

AVB (figure 28) is a circular arc.

I = AI.IB = CA.CB is the tangential deflection, or angle of intersection, of the arc.

s is the number of stations in the arc.

K = ½ I is the *chord deflection*, or deflection from the tangent, AI, to the chord, AB.

D is the degree of curvature, or tangential deflection over an arc with a chord of 100 feet.

r = (**CA**) is the radius of curvature.

k = (**AB**) is the length of the chord.

l is the length of the arc.

t = (**AI**) = (**IB**) is the semi-tangent.

Sec. ½ I = (**CI**) is the secant of the half arc.

Ex. sec. ½ I = (**VI**) is the external secant of the half arc.

Versin. ½ I = (**PV**) is the versed sine of the half arc.

e = (**TE**) is the tangential ordinate of the (arc AE = Ie), at a distance, (**AT**) = n, from the initial tangent-point at A.

$o_{½}$ = (**PY**) = versin. ½ I is the middle ordinate, or ordinate of the chord, at half its length from A.

(2) When the angle, I, or D, is a factor in any of the following equations, it should be expressed in degrees and decimals, according to the rules of problem I.

I = s.D. (1)

Sin. ½ I = ½ k ÷ r = k.sin. ½ D ÷ 100. (2)

Cos. ½ I = ½ k ÷ t. (3)

Tan. ½ I = t ÷ r = 2 t.sin. ½ D ÷ 100. (4)

Sin. ½ D = 100 ÷ 2r = 100 sin. ½ I ÷ i = 100.tan. ½ I ÷ 2 t. (5)

Versin. I = r (1 − cos. I) = 2 r.sin.^{2} ½ I = 100 sin.^{2} ½ I ÷ sin. ½ D. (6)

Vers. (180° − I) = r (1 + cos. I = 2 r.cos.^{2} ½ I = 100.cos.^{2} ½ I ÷ sin. ½ D. (7)

Sec. ½ I = r ÷ cos. ½ I = t ÷ sin. ½ I. (8)

Ex. sec. I = sec. I − r. (9)

r = 50 ÷ sin. ½ D = ½ k ÷ sin. ½ I = t ÷ tan. ½ I. (10)

k = 2 r.sin. ½ I = 100 sin. ½ I ÷ sin. ½ D = 2 t.cos. ½ I. (11)

l = r.I × 0.01745 = I × 0.87266 ÷ sin. ½ D. (12)

Log. of 0.01745 = 8.241877. Log. of 0.87266 = 9.940847.

t = 50 tan. ½ 1 ÷ sin. ½ D = r.tan. ½ I. (13)

Sin. I$e = n \div r = 2\,n.\sin.\ \frac{1}{2}\,D \div 100.$ (14)

$e = r.\text{versin. I}e = n.\tan.\ \frac{1}{2}\,Ie = 50$ vers. I$e \div \sin.\ \frac{1}{2}\,D.$

$o_{\frac{1}{2}} = r.\text{versin.}\ \frac{1}{2}\,I = 100.\text{versin.}\ \frac{1}{2}\,I \div 2\ \sin.\ \frac{1}{2}\,D = \frac{1}{2}\,k.\tan.$ $\frac{1}{2}\,I =$ tang. ordinate for an abscissa equal to $\frac{1}{2}\,k.$ (15)

The ordinate at a distance, d, from the middle ordinate, is equal to the middle ordinate, $o_{\frac{1}{2}}$, minus the tangential ordinate, for an abscissa equal to d. (16)

3. *Tangential ordinates*, calculated by (14), for abscissas of from 5 to 100 feet, and for curvatures of from 1 to 15 degrees, are given in table 1.

4. *Chord Ordinates.*—Middle ordinates may be calculated by (15) from table 1, also the ordinates at any distance, d, from the middle ordinate, if the chord, k, be calculated by (11).

Example. Required the middle ordinate of a 10-degree curve, with a chord, k, equal to 100′, also the ordinates at 25′, 50′, 75′, and 100′ from the middle ordinate (see figure 32).

Middle ordinate, $o_{\frac{1}{2}} =$ tan. ord., e, for an abscissa, n, of 50′ $= 2.18.$

Ord. for $(d=25') = o_{\frac{1}{2}} -$ tan. ord. for $25' = 2.18 - .55 = 1.63.$

Ord. for $(d=50') = o_{\frac{1}{2}} -$ tan. ord. for $50' = 2.18 - 2.18 = 0.00.$

Ord. for $(d=75') = o_{\frac{1}{2}} -$ tan. ord. for $75' = 2.18 - 4.92 = -2.74.$

Ord. for $(d=100') = o_{\frac{1}{2}} -$ tan. ord. for $100' = 2.18 - 8.78 = -6.60.$

The two last ordinates are negative, and outside of the arc, and extend, from the chord, in an opposite direction to the middle ordinate.

PROBLEM XIV.

TANGENTIAL, CHORD, AND CURVE ORDINATES.

Approximate Formulæ.

Tangential ordinates to curves are measured at right angles to the tangent; *chord ordinates*, at right angles to the chord; and *curve ordinates*, from one curve to another, at right angles to the tangent of one of the curves.

For short chords, and angles such as occur in turn-outs from railroad-tracks, these ordinates may be determined by simple formulæ, which, though not exact, suffice for all cases rising in practice.

The formulæ of this problem are based upon the following propositions, of which all, except I. and II., are applicable only within certain limits of the chord and angles, which are stated in the consideration of the different formulæ:

I. *The middle ordinate* of an arc is equal to the tangential ordinate for an abscissa equal to half the chord of the arc.

II. The chord ordinate at a distance, *d*, from the middle ordinate, is equal to the middle ordinate, less the tangential ordinate for an abscissa equal to *d*.

III. *Tangential ordinates* vary in direct proportion to the degree of the curve and the square of the abscissa.

IV. The sum or difference of the tangential ordinates of two curves, with degrees, D*a* and D*b*, is equal to the tangential ordinate of a curve with a degree equal to the sum or difference of D*a* and D*b*, the abscissa being the same in each case. The same rule will apply to ordinates from chords of equal length.

V. For small angles the cosines may be assumed equal to unity, and the sines and tangents to increase in direct proportion to the angles; and for angles not exceeding 20 degrees the sine of the half angle, ½ I, may be expressed in terms of the angle, I, by the following arithmetic equation:

$$\text{Sin. } \tfrac{1}{2}\text{I} = \text{I} \div (114.4 + .04\ \text{I}). \quad (1)$$

2. *The tangential ordinates* in table 1 are calculated by (14), problem XIII., and will serve to check the accuracy of the result obtained from the approximate formulæ.

(Figures 29, 30, and 31.) T is the tangent-point, or t. p.; (**TE**) = *n*, the common abscissa of the two curves TA and TB; I*e*, the angle of the curve; D*a* and D*b*, their degrees of curvature, and e_A and e_B their ordinates at the distance, *n*, from the t. p.

The curves and ordinates are in opposite directions in figure 30, and in the same direction in figure 31, and, assuming the degree and ordinate of the curve through B to be positive, the degree and ordinate of the curve through A will be negative in figure 30, and positive in figure 31.

$$e = .88 . \text{D} . (n \div 100)^2 = .88 . \text{I}e . (n \div 100). \quad (2)$$

$(e_B - e_A)$ = ordinate for D = (D*b* − D*a*), = .88 . (D*b* − D*a*) . $(n \div 100)^2$. (3)

When the main track is curved, and D*a* the degree of curvature, the middle ordinate, $o'_{\frac{1}{2}}$, of the turn-out curve, is equal to the middle ordinate, $o_{\frac{1}{2}}$, from (7), for a straight track, plus or minus the middle ordinate of the main-track rail, from (5), and therefore,

$$o'_{\frac{1}{2}} = \begin{pmatrix} o_{\frac{1}{2}} \pm .29 \times Da \times [i + (\mathbf{F} + \mathbf{S})]^2 \\ o_{\frac{1}{2}} \pm .218 \times Da \times (k + 100)^2 \end{pmatrix} = .218 \times k + 100 \times (\mathbf{F} - \mathbf{S} \pm Da \times k + 100). \quad (7a)$$

For the double sign in this equation use the plus sign for an inside, and the minus sign for an outside, turn-out. The side ordinates will be three-fourths of the middle ordinate.

Example. Turn-out from a standard-gauge track: $i = (g - s) = 4.29$. $F = 7°$; $S = 1°\ 20' = 1.33°$. $(F - S) = 5°\ 40' = 5.67°$; $(F + S) = 8°\ 20' = 8.33°$.

If the main track is straight:

(7) $o_{\frac{1}{2}} = \frac{1}{4} i . (F - S) + (F + S) = (1.07) . (5.67) + (8.33) = 0.73$.

If the main track is curved, and the degree, D*a*, is $8°\ 22' = 8.37°$:

(7) $o_{\frac{1}{2}} = .73$ is the middle ordinate for a straight main track.

(5) $.29 \times Da \times [i + (\mathbf{F} + \mathbf{S})]^2 = .29 \times 8.37 \times (4.29 + 8.33)^2 = 0.64$, is the middle ordinate of the main-track rail.

(7a) $o'_{\frac{1}{2}} = \begin{cases} 0.73 + 0.64 = 1.37 \text{ for an inside turn-out.} \\ 0.73 - 0.64 = 0.09 \text{ for an outside turn-out.} \end{cases}$

These formulæ apply to three-throw switches by making $S = 2S$; to sidings by making $S = 0$, and to crossings by making the greater frog-angle equal to F, and the lesser to S, taking the values of i from table IX. problem XVIII. The middle ordinate of a rail of length, l, is equal to $o'_{\frac{1}{2}} \times (l + k)^2$; and, by the use of these formulæ, it will never be necessary to determine the degree, Db, of the turn-out curve for the purpose of bending the rail or laying off the curve.

4. *Curve Ordinates* (figures 33 to 35).—In figure 33, the ordinates of the arc BF, from the chord, **BF**, may be determined in terms of F, S, and the distance, $(\mathbf{AB}) = i$. If from Op, on the chord (**BF**), a perpendicular be let fall on **AI** at Qp, the distance ($\mathbf{Q}p\,\mathbf{O}p$) is equal to $(1 - p) . i$, and the distance from **AI** to the arc BF, on the line $\mathbf{Q}p\,\mathbf{O}p$, is equal to the distance

from **AI** to the chord **BF**, plus the distance from that chord to the arc, which latter distance may, without appreciable error, be assumed equal to the chord ordinate o_p.

Representing the distance from Qp to the arc BF by the symbol q_p, and taking the value of o_p from (6), the value of q_p is expressed by the following equation:

$$q_p=(1-p).i+o_p=(1-p).(1+p).i-p.(1-p)\times 1.75.S.k\div 100. \quad (8)$$

Making p equal to $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$, the side and middle *curve ordinates* will be as follows:

$$\left.\begin{aligned} q_{\frac{1}{4}} &= \tfrac{15}{16}i-.33\times S.k\div 100=\tfrac{3}{4}i\left(1+\tfrac{1}{4}\frac{F-S}{F+S}\right)\\ q_{\frac{1}{2}} &= \tfrac{12}{16}i-.44\times S.k\div 100=\tfrac{1}{2}i\left(1+\tfrac{1}{2}\frac{F-S}{F+S}\right)\\ q_{\frac{3}{4}} &= \tfrac{7}{16}i-.33\times S.k\div 100=\tfrac{1}{4}i\left(1+\tfrac{3}{4}\frac{F-S}{F+S}\right)\end{aligned}\right\} \quad (9)$$

These are the ordinates to the arc BF from the straight curve **AI**, for which the degree of curvature is zero. *They are also the curve ordinates for any pair of curves, for which the difference of the degrees of curvature is equal to the degree of curvature of the arc* BF, *in figure* 33, in which the degree of the curve through A is equal to zero. It is shown in proposition XV. that such is the case with each pair of curves through A and B, intersecting at the same angle, F, and therefore the formula will apply to the determination of the ordinates between the intersecting rails of turn-outs from railroad-tracks, when the chord distance, k, to their intersection, and the switch-angle S, are known, or the frog and switch-angles, F and S.

When these curve ordinates are known, the turn-out rail may be laid down by ordinates from the main-track rail, and the ordinates of the turn-out rail determined, without taking into consideration the degrees of curvature or whether the turn-out is to the inside or outside.

Example 5. (**AB**)$=i=4.29$ is the distance, at the toe of the switch, between the main-track and turn-out rails; S$=1°\ 20'$; the angle, F, at which the two curves intersect, is $7°\ 00'$; the degree, Da, of the main-track curve is $10°\ 00'$, and the turn-out

is to the inside. By exact formulæ the degree, D*b*, of the turn-out curve is 19° 36′, and the distance between the intersecting rails, measured at right angles to the tangent line through A, at half the chord distance to the intersection, F, is equal to 2.87. The distance by (9) is 2.86.

The formulæ of this problem are placed here for reference in the solution of succeeding problems, and will be intelligible only when the problems to which they refer are understood. For example of the application see table IX. problem XVIII.

PROBLEM XV.

COMPOUND ARCS, WITH REGULARLY INCREASING DEGREES OF CURVATURE.

In table XI. (A, B, and C), are shown alignments of curves, with degrees of curvature increasing at the rates of 1, 2, and 3 degrees per quarter station.

These tables are calculated, as in example 4, problem XI., by departures and latitudes, and give, opposite to each quarter station, the *short chord* from previous station, which is very nearly 25 feet; *the deflection*, K*o*, from the initial tangent to the *long chord* ending at the station; the length, *k*, of the *long chord; the deflection*, K*z*, from the end of the long chord to the tangent at the terminal station; the *total tangential deflection*, I*o* = K*o* + K*z*, over the compound arc, and the *offsets* along, and at right angles to, the tangent through station 0.

The curves may be run in by short chords of 25 feet, as in the case of circular curves, but the deflections, K*o* and K*z*, at the beginning and end of a long chord, are not equal.

It will be more convenient, as a rule, to lay off, first, the extremities of the set of compound curves, and the intermediate stations may then be put in, when convenient, either by chord deflections or by offsets from the tangent. The angles are given to fractions of a minute, and the distances to hundredths of a foot, but it will be sufficiently accurate to take the nearest minute and tenth.

Example 1. To change from a tangent to a 10-degree curve by curvatures increasing at the rate of two degrees to the quar-

ter station, as in table B, deflect $Ko = 1° 52'$ to the long chord ending at the E. C. C. 8°, at station 1.00; measure the chord, $k = 100$, and from its extremity deflect, $Kz = 3° 08'$, to the terminal tangent through station 1.

To change from a 10-degree curve to a tangent, measure the deflection, $Kz = 3° 08'$, to the long chord, from 1 to 0, and the deflection, $Ko = 1° 52'$, from that chord to the tangent at station 0.

In either case the deflection from the initial to the terminal tangent will be $Io = Ko + Kz = 5° 00'$.

To change from a 4 to a 10 degree curve, it would be necessary to run *back* to the beginning of the 2-degree curve at station 0, and then run in the curves, from 4 to 10 degrees, from the initial tangent-point at station 0.

The change of curvature may be made by curves with degrees increasing by one, two, or three degrees per quarter station, but the distance by table A will be three times the distance by table C.

3. *Example* 1 (figure 37). To substitute, at each t. p. of a located curve, a compound arc, and an intermediate, or middle, arc between their extremities.

Case 1. *When the initial point of the compound arc is at the extremity of the located curve.*

k is the *long chord* to the terminal point of the compound arc.

Ko and Kz are the deflections to, and from, the long chord, from the initial to the terminal tangent of the compound arc, and $I = Ko + Kz$ is the total tangential deflection over the arc.

n and e are the offsets to the extremity of the *long chord*, from the tangent through its initial point.

I' is the angle, and D' the degree of curvature, of the located curve.

$i' = 100 . \sin. \frac{1}{2} I' \div \sin. \frac{1}{2} D'$ is the chord of the located curve.

$i'' = i' - 2k . \cos. (\frac{1}{2} I' + Ko)$ is the chord of the middle arc.

$I'' = I' \div 2 (Ko + Kz)$ is the angle of the middle arc.

$\sin. \frac{1}{2} D'' = 100 . \sin. \frac{1}{2} I'' \div i''$ gives the degree of curvature of the middle arc, and $s'' = I'' \div D''$ the number of stations. For example, $D' = 10° 00'$ is the degree, and $I' = 120° 00'$ is the angle, of the located curve and it is required to substitute at each extremity of curve a compound arc ending at E. C. C.

8°, and to determine the corresponding *middle arc* between the extremities of the compound arcs.

By table B, the length of the chord terminating at station 1.00 is 100.00; the deflection, K*o*, to the chord, is 1° 52′ 30″; the deflection, K*z*, *from* the chord *to* the tangent at station 1 is 3° 07′ 30″, and the tangential deflection over the compound arc is therefore equal to K*o* + K*z* = 5° 00′.

$i' = 100.\sin. \frac{1}{2} I' \div \sin. \frac{1}{2} D' = 993.7$ is the chord of the *located arc.*

$i'' = i' - 2k.\cos. (\frac{1}{2} I' - Ko) = 888.0$ is the chord of the *middle arc.*

Sin. $(\frac{1}{2} D'' = 5° 17' 30'') = (100. \sin. \frac{1}{2} I'' \div i'')$ is the sine of half the degree, $(D'' = 10° 35')$, of the middle arc, and $s'' = (I'' \div D'')$, equal to 10.39, is the number of stations.

The alignment to be substituted is therefore as follows:

One station of *compound arc,* tangential deflection 5° 00′.

10.39 stations of a 10° 35′ *middle arc,* tangential deflection 110° 00′.

One station of compound arc, tangential deflection 5° 00′.

The number of stations is, therefore, 12.39, and the alignment is lengthened 39 feet.

Case 2. In the preceding example, the deflection, in 100 feet, was equal to 10 degrees over the 10-degree curve, and to 5 degrees over the compound arc, and the loss of 10 degrees in the two compound arcs increased the degree of curvature of the *middle arc* by thirty-five minutes, and for a shorter located curve the increase would have been still greater. If, however, there should be a sufficient length of tangent at the extremities of the located curve, this increase of the degree of curvature may be obviated by commencing the compound arc a distance *back* on the tangent, to be determined in the following manner:

The deflection, K*o* + K*a*, over the compound arc, is 5 degrees in a distance, *k*, of 100 feet, while the same deflection is made on the 10-degree curve in 50 feet. Therefore, if the compound arc be commenced at a distance, $[k = (Ko + Kz).100 \div D' = 50]$ feet back of the t. p., the tangent of the compound arc at station 1.00 will be parallel to the tangent of the located curve 50 feet beyond the t. p. and the relative position of the parallel tangent-points may be determined from their tangential ordinates in the following manner:

The offset to station 1.00 of the compound arc, at 100 feet from the initial point, which is 50 feet from the t. p. of the located curve, is 3.27.

The tangential offset to the 10-degree curve, at 50 feet from the t. p., is 2.18, and therefore the parallel tangent-points are 1.09 feet apart, measured at right angles to the tangent through the initial point of the compound arc, station 1.00 of that arc being on the inside.

As, therefore, the curves are 1.09 feet apart, on a line making an angle of 5 degrees with their radial directions, the radius of the middle arc may be assumed to differ but one foot from the radius of the 10-degree curve, with which it therefore practically coincides.

PROBLEM XVI.

INTERSECTING CURVES—FIGURES 47 TO 63.

(*See table III. for formulæ, and proposition XIV. for demonstration.*)

Two curves from tangent-points, A and B, intersect at F. a is the radius, and Da the degree of curvature, of the curve through A, and Xa is the tangential deflection, over that curve, from A to F.

b is the radius, and Db the degree of curvature, of the curve through B, and Xb is the tangential deflection over that curve from F to B.

F is the deflection *from* the terminal tangential direction of the arc AF *to* the initial tangential direction of the arc FB, and the algebraic sum, Xa + F + Xb, of these three deflections, is equal to the deflection I *from* the tangent through A *to* the tangent through B.

(**AB**) = i is the distance between the points A and B, and A′ is the greater, and B′ the lesser, of the acute angles between **AB** and the tangents through A and B.

I is the deflection *from* the tangent through A *to* the tangent through B, *over the curves produced to contact with their*

common tangent, and is determined by the rules of problem V. in the following manner: I will be equal either to (A′+B′), or to 180°−(A′+B′), when the tangents intersect as in figure 7, and to (A′−B′), or to 180°−(A′−B′), when they intersect as in figure 7a. By problem V., I will be equal to (A′ ± B′) when the centres of the curves are on the same side of **AB**, and to 180°−(A′ ± B′) when they are on opposite sides.

The deflection, F, from the terminal tangent of the arc AF to the initial tangent of the arc FB may be so made as to be equal either to the acute angle ±F′ or to the obtuse angle ∓(180°−F′), but in the application of this problem F *must have the value which will make the algebraic sum*, (X*a*+F+X*b*), *equal to* I, as more fully explained in problem VI. The values of I and F, for different forms of curves, are given in the formulæ of table III., and also in problem VI.

The only difficulty in the application of the formulæ is to determine the sign of F, and whether it is acute and equal to F′, or obtuse and equal to 180°−F′; and both of these points may be determined by inspection of the figure, according to the rules of problem VI. In this connection, it is also to be observed that the curves intersect at two points, corresponding to the positions of the pair of equal tangents, and therefore, while a change of the sign of F in the formulæ will change the values of X*a* and X*b*, it will not affect the values of the radii. In figure 49, for instance, the radius, *b*, is infinite; the curve through B coincides with the tangent **BI** and the deflection X*b* = 0.

Estimating the deflections from the tangential direction AI, and the right as positive, for the right-hand crossing I = AI.BI = −(A′−B′); the deflection, X*a*, from AI to the terminal tangent at F, is to the *right*, and the deflection *from* that tangent *to* BI is to the *left*, and therefore X*a* − F + X*b* = −(A′−B′), and X*a* + X*b* = X = (F−I).

For the left-hand crossing X*a* and F are to the *left*, and therefore − X*a* − F = −I, and X*a* = (I + F).

The angle between the chords **AF** and (**FB**), as explained in problem VIII., will be ½ (I+F) for the right, and ½ (I−F) for the left-hand crossing; and as the same rule will apply, whatever may be the radii of the curves, the solution of the problem is based upon the following propositions.

Let (X = X*a* + X*b*) be the *algebraic* sum of the deflections

over the *arcs* AF and FB, and K the angle between the *chords* **AF** and **FB.**

$$X = Xa + Xb = (I - F). \quad (1) \qquad K = \tfrac{1}{2}(I + F). \quad (2)$$

This difference or sum is *algebraic*, and the sign of F may be changed, *but the signs in* (1) *and* (2) *will be always different.*

The formulæ for sidings, crossings, and turn-outs from railroad-tracks are deduced from the formulæ of this problem by making I equal to the switch-angle, and F to the frog-angle; and the formulæ for compound and reversed curves, in table X., problem XXII., by making F equal to zero or to 180°.

2. *The solution by construction* is based on the following proposition, which is exemplified in figure 47:

If F *is the intersection of any pair of curves through* A *and* B; I *the deflection between the tangents through* A *and* B, *intersecting at* I; *and* F *the deflection between the tangents crossing at* F, *the intersection of the curves will be on the circumference of a circle,* ABF, *of which the radius, r, is equal to* $\frac{1}{2}\, i \div \sin. \frac{1}{2}(I + F)$.

The angle $\frac{1}{2}(I + F)$ is the angle between the chords, and is determined by the rules of the preceding article and by formula (2); and if O is the centre, and r the radius, of the circle ABF, and $o = (\mathbf{MO})$ the perpendicular distance to O from the middle of (**AB**),

$$r = \tfrac{1}{2}\, i \div \sin. \tfrac{1}{2}(I + F). \quad (3)$$
$$o = \tfrac{1}{2}\, i . \cot. \tfrac{1}{2}(I + F) = r . \cos. \tfrac{1}{2}(I + F). \quad (4)$$

If, therefore, one of the curves, as AF, is located, its intersection by a second curve, BF, at an angle F, is determined by measuring, from the middle of (**AB**) to O, the perpendicular distance o, and from O, as a centre, cutting the located curve with a radius r. This rule will apply to any pair of crossing or turn-out curves, and also to compound and reversed curves, by making F equal to 0 or 180°, noting that when $(I + F)$ is equal to $180° - (A' \pm B' + F)$, the sine of $\frac{1}{2}(I + F)$ is equal to cos. $\frac{1}{2}(A \pm B' + F)$.

In the application of these formulæ to turn-outs from railroad-tracks, the curves should be those of the intersecting rails; or if the centre lines of the tracks are to be laid off, the tangent-point should be at a distance of half the gauge from the frog-point, F.

For application to the construction of diagrams for turn-outs from switches, see problem XVIII., art. 2 (Pl. 14).

Example 1 (figure 47). **AF***a* and **F***b***B** are the tangents to a main track and turn-out for a car-house, of which B is the entrance. (**AB**) $= i = 100'$; BAFa $= A' = 90°$; ABFb $= B' = 85°$, and the turn-out is to be made with a No. 7 frog (angle $8° 10' = F$).

$I = (A' - B') = 90° - 85° = 5° 00'$.

$r = \frac{1}{2}\, i \div \sin. \frac{1}{2}\,(I + F) = 50 \div \sin. 6° 35' = 436.1$.

$o = \frac{1}{2}\, i . \cot. \frac{1}{2}\,(I + F) = 50 . \cot. 6° 35' = 433.2$.

It will be seen that the values of r and o are independent of the values of the radii, and of the direction of the turn-out, whether to inside or outside.

From BF, when determined, the tracks may be laid off for other tracks in the same manner, observing that, if the tangents through A and B are parallel, and at right angles to **AB**, I is equal to zero, and r to $\frac{1}{2}\, i \div \sin. \frac{1}{2}\,F$.

In the cases of crossings and turn-outs from switches, r may be measured with a tape-line; but if, as in the case under consideration, the distance, r, is too great to be described by a tape or chain, measure the distance, $2 \times o$, along AF, to a point, F, and also the distance (**OF**). If this distance, (**OF**), is not equal to r, change the position of F to make it equal.

3. *Formulæ* (figure 47). Two curves, through A and B, with radii, a and b, intersect on the circumference of the circle AFB, with radius, $r = \frac{1}{2} i \div \sin. \frac{1}{2}\,(I + F)$.

The sides of the inscribed triangle AFB are, $k_a = (\mathbf{AF}) =$ chord of arc AF; $k_b = (\mathbf{FB}) =$chord of arc BF, and $i = (\mathbf{AB})$ = distance from A to B.

As $\frac{1}{2}\, Xa$ is the deflection from tangent AI to chord **AF**, and $\frac{1}{2}\, Xb$ the deflection from tangent **BI** to chord **BF**, if A and B are the interior angles of the triangle AIB, the angles of the triangle AFB will be:

$$\left.\begin{array}{l} \text{BAF} = A + \frac{1}{2} Xa, \text{ when curve through A is convex to } \mathbf{AB}. \\ \quad\text{“}\quad A - \frac{1}{2} Xa, \quad\text{“}\quad\quad\text{“}\quad\quad\text{“}\quad\text{concave}\quad\text{“} \\ \text{ABF} = B + Xb, \text{ when curve through B is convex to } \mathbf{AB}. \\ \quad\text{“}\quad B - Xb, \quad\text{“}\quad\quad\text{“}\quad\quad\text{“}\quad\text{concave}\quad\text{“} \\ \text{AFB} = K = \frac{1}{2}(I + F). \end{array}\right\} \quad (5)$$

Oa and Ob are the angles, at F, between the tangent of the ircle AFB and the tangents of the arcs AF and BF. These

angles are constant for every intersection of the arcs on the circumference AFB, whatever may be the values of the radii, a and b, and Oa is therefore equal to the angle between the tangent of the circle AFB and the straight arc **AI**, for which the radius, a, is infinite; and Ob to the angle between the tangent of the circle AFB, and the straight arc, **BI**, with radius, b, equal to infinity.

The three angles, K, Oa, and Ob, are each equal to the sums or differences of the angles A′, B′, and F, and differ only in the signs of the terms, which may be determined by the following equations, observing that A′ represents the greater of the two acute angles A′ and B′, and that ½ (I + F) is the angle, K, between the chords, determined by (2).

$$\left.\begin{array}{l} \tfrac{1}{2}(\mathrm{I}+\mathrm{F}) = \mathrm{K} = \tfrac{1}{2}(\mathrm{A}' \pm \mathrm{B}' \pm \mathrm{F}) \text{ when } \mathrm{I} = (\mathrm{A}' \pm \mathrm{B}'). \\ \tfrac{1}{2}(\mathrm{I}+\mathrm{F}) = \mathrm{K} = 90^\circ - \tfrac{1}{2}(\mathrm{A}' \pm \mathrm{B}' \pm \mathrm{F}) \text{ when } \mathrm{I} = 180^\circ - (\mathrm{A}' \pm \mathrm{B}'). \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l} \mathrm{O}a, \text{ or } 180^\circ - \mathrm{O}a, = \tfrac{1}{2}(\mathrm{A}' \mp \mathrm{B}' \pm \mathrm{F}) \text{ when } \mathrm{I} = (\mathrm{A}' \pm \mathrm{B}'). \\ \mathrm{O}a, \text{ or } 180^\circ - \mathrm{O}a, = 90^\circ - \tfrac{1}{2}(\mathrm{A}' \mp \mathrm{B}' \pm \mathrm{F}) \text{ when } \mathrm{I} = 180^\circ - (\mathrm{A}' \pm \mathrm{B}'). \end{array}\right\} \quad (7)$$

$$\left.\begin{array}{l} \mathrm{O}b, \text{ or } 180^\circ - \mathrm{O}b, = \tfrac{1}{2}(\mathrm{A}' \mp \mathrm{B}' \mp \mathrm{F}) \text{ when } \mathrm{I} = (\mathrm{A}' \pm \mathrm{B}'). \\ \mathrm{O}b, \text{ or } 180^\circ - \mathrm{O}b, = 90^\circ - \tfrac{1}{2}(\mathrm{A}' \mp \mathrm{B}' \mp \mathrm{F}) \text{ when } \mathrm{I} = 180^\circ - (\mathrm{A}' \pm \mathrm{B}'). \end{array}\right\} \quad (8)$$

It will be seen that Ob differs from ½ (I + F) in the signs of B′ and F, and from Oa in the sign of F only. (7) and (8) do not determine whether the second members of the equations are equal to Oa and Ob, or to their supplements.

From these relations between the sides and angles of the triangle AFB the following formulæ are deduced:

$$\left.\begin{array}{l} \text{cot. } \tfrac{1}{2}\mathrm{X}a = a \div r \text{ . sin. O}a \pm \text{cot. O}a\text{.} \\ \text{cot. } \tfrac{1}{2}\mathrm{X}b = b \div r \text{ . sin. O}b \pm \text{cot. O}b\text{.} \end{array}\right\} \quad (9)$$

$$\tfrac{1}{2}\mathrm{X}b = (\mathrm{I} - \mathrm{F}) \mp \tfrac{1}{2}\mathrm{X}a. \qquad \tfrac{1}{2}\mathrm{X}a = (\mathrm{I} - \mathrm{F}) \mp \tfrac{1}{2}\mathrm{X}b. \quad (10)$$

$$\left.\begin{array}{l} b = r\text{.sin. } (\mathrm{A}' \pm \tfrac{1}{2}\mathrm{X}a) \div \text{sin. } \tfrac{1}{2}\mathrm{X}b. \quad a = r\text{.sin. } (\mathrm{B}' \pm \tfrac{1}{2}\mathrm{X}b) \\ \qquad + \text{ sin. } \tfrac{1}{2}\mathrm{X}a. \end{array}\right\} \quad (11)$$

$$\left.\begin{array}{l} k_a = 2\,a\text{ . sin. } \tfrac{1}{2}\mathrm{X}a = 2\,r\text{ . sin. } (\mathrm{B}' \pm \tfrac{1}{2}\mathrm{X}b). \\ k_b = 2\,b\text{ . sin. } \tfrac{1}{2}\mathrm{X}b = 2\,r\text{ . sin. } (\mathrm{A}' \pm \tfrac{1}{2}\mathrm{X}a). \end{array}\right\} \quad (12)$$

In (9), cot. Oa or cot. Ob is + for convex curves through A or B, and − for concave curves.

In (10), the signs can be obtained from the algebraic sum, Xa + F + Xb = I, determined, for the particular curves under consideration, by the rules of problem VI. The signs of F in (6) and (10) will be different.

In (11), Xa or Xb will be + for convex curves through A or B, and − for concave curves.

To these rules there are exceptions, explained in proposition XIV.

The proper signs for such cases as are likely to occur in practice are given in the formulæ of table III., in terms of the acute angles A′, B′, and F′, with reference to a corresponding figure in each case. The terms in these formulæ are so arranged as to give positive results for the form of curves represented in the figure to which reference is made. Should the result be negative, adopt the formula giving positive results for the given quantities, or change the values of the quantities so that the result may be positive.

4. *Formulæ deduced from Problem XXVI.*—The relations between the values of the radii, a and b, are also expressed by the following formulæ, derived from (1), problem XXVI. (see article 11, proposition XVI.), by making t equal to zero, and substituting (vers. I − vers. F) for vers. I. The values of the angles and chords might also be obtained from the formulæ of problem XXIV., but may be better determined by the formulæ of the preceding article.

$$\left.\begin{aligned} a&=(+\tfrac{1}{2}i\pm b.\text{sin. B}')\div[\mp\text{sin. A}'-(\text{vers. I}-\text{vers. F}).b\div i]\\ b&=(+\tfrac{1}{2}i\pm a.\text{sin. A}')\div[\mp\text{sin. B}'-(\text{vers. I}-\text{vers. F}).a\div i]\end{aligned}\right\}\quad(13)$$

The signs of the terms in these equations are determined by the following rules:

For the double sign prefixed to a term containing sin. A′ or sin. B′, substitute the upper sign when the curve through A or B is convex to **AB**, and the lower when it is concave.

The sign prefixed to (vers. I − vers. F) is minus, but when vers. F is greater than vers. I the difference is negative, and the sign prefixed must be changed.

For (vers. I − vers. F) may be substituted (cos. F − cos. I), but it is to be noted that the cosine of an obtuse angle is negative.

The values of F in any particular case, whether F′, or 180° −F′, may be taken from table III., observing that vers. (180°− F′) is equal to (1+cos. F′), and vers. [180°−(A′ ± B′)] to [1 + cos. (A′ ± B′)]. See formula (7), problem XIII.

The solution will be possible for all values of the radii which give like signs to the numerator and denominator, and impossible for values giving unlike signs.

In the following equations, the signs of *all the terms* are reversed, from the rule given above, to provide for the cases in which the departure, a sin. A′, or b sin. B′, is greater than $\frac{1}{2}i$, as in the case of turn-outs from railroad-tracks.

When A′ = B′ = 90°, I = 0, and one curve will be concave, and the other convex, when the centres are on the same side of **AB**, as in the case of a turn-out to the inside (figure 50¾); and both curves concave when the centres are on different sides, as in the case of a turn-out to the outside (figure 50½); and if b is the concave curve, in both cases,

$$\left.\begin{aligned} a &= (+b - \tfrac{1}{2}i) \div (\pm 1 \mp \text{vers. F}.b \div i). \\ b &= (+a \pm \tfrac{1}{2}i) \div (\pm 1 \mp \text{vers. F}.a \div i). \end{aligned}\right\} \quad (14)$$

Use upper sign when the centres are on the same side of **AB**.
" lower " " " different sides "

When one of the curves, as that through A, *is a straight line,* the radius, a, is infinite, and the denominator of (13a) must, therefore, be infinite, giving the following value for b:

$$b = i \text{ sin. A}' \div (\text{vers. I} - \text{vers. F}) = i \text{ sin. A}' \div (\cos. \text{F} - \cos. \text{I}). \quad (15)$$

If also, as in (14), A′ = B′ = 90°,

$$b = i \div \text{vers. F}. \quad (16)$$

When a and b are given, F *may be determined by the following formula,* derived from (13):

$$\left.\begin{aligned} \text{Vers. F} &= \text{vers. I} + i(+\tfrac{1}{2}i \pm a.\text{sin. A}' \pm b.\text{sin. B}') \div a.b. \\ \text{Cos. F} &= \cos. \text{I} - i\,(+\tfrac{1}{2}i \pm a.\text{sin. A}' \pm b.\text{sin. B}') \div a.b. \end{aligned}\right\} \quad (17)$$

The signs of the terms in these equations are to be determined as in (13), viz.:

a sin. A, or b sin. B, is + for convex curves through A or B.
" " " − " concave " " "

When $I = 180° - (A' \pm B')$, cos. $I = -$ cos. $(A' \pm B')$, and vers. $I = 1 +$ cos. $(A' \pm B')$; and, conversely, when the resulting value of vers. F is greater than unity, or when the cosine is negative, F will be obtuse, and equal to $180° - F'$.

(Figure 91.) When $A' = B' = I = 0$, (**AB**) is the common tangent to the two circles, and Xa and Xb are the tangential deflections from its extremities to the intersection, F, of the arcs.

Representing this value of i by t, and Xa and Xb by Fa and Fb, the following equations may be derived from (17) and (9):

$$\tfrac{1}{2}\, t = \text{sin. } \tfrac{1}{2}\, F. \sqrt{ab}. \quad \text{cot. } \tfrac{1}{2}\, Fa = a + \tfrac{1}{2}\, t + \text{cot. } \tfrac{1}{2}\, F.$$
$$\text{cot. } \tfrac{1}{2}\, Fb = b + \tfrac{1}{2}\, t + \text{cot. } \tfrac{1}{2}\, F. \quad (18)$$

5. *Examples.* AF (figure 51¼) is the curve of the rail of a railroad-track, and B a tangent-point on the rail of another track, which is to be connected with the rail AF by means of a frog of angle $(F = 7°)$; AF is a 2-degree curve, (rad. $a = 2{,}865$), $(\mathbf{AB}) = i = 50$; angle $BAI = A' = 86° 00'$; angle $ABI = B' = 76° 00'$; curve through A convex, and curve through B concave, to **AB**.

It is required to find the radius, b, of a curve which will intersect the rail AF at the given frog-angle, and also the distance, $(\mathbf{AF}) = k_a$, from the point A to the frog.

As the triangle AIB, formed by the tangents through A and B and the line **AB**, has one of the interior angles obtuse, by problem V. I must be equal either to $(A' - B')$ or to $180° - (A' - B')$, and, as the centres of the curves are on the same side of **AB**, it will be equal to $(A' - B')$. The radius, b, and the chord distance, k_a, may therefore be calculated, as follows, from the formulæ of table III. for figure 51¼:

Radius, $(a = 2{,}865)$.....			log.	3.457115
Sin. $[\frac{1}{2}\,[A' - (B' - F)] = 8° 30']$....			"	9.169702
				2.626817
$\frac{1}{2}\,(\mathbf{AB}) = \frac{1}{2}\, i = 25$.....	log.	1.397940	"	
Sin. $[\frac{1}{2}\,[A' + (B' - F)] = 77° 30']$...	"	9.989582	"	1.387522
17.350.....			"	1.239295

Cot. ($\frac{1}{2}$ Xa = 3° 15′) = 17.350 + cot. 77° 30′ = 17.572.
($\frac{1}{2}$ Xb = 1° 45′) = $\frac{1}{2}$ Xa − $\frac{1}{2}$ [A′ − (B′ + F)] =
3° 15′ − $\frac{1}{2}$ (86° 00′ − 83° 00′).

$\frac{1}{2}$ i = 25′			log.	1.397940
Sin. (A′ − $\frac{1}{2}$ Xa = 82° 45′)			"	9.996514
			"	1.394454
Sin. ($\frac{1}{2}$Xb = 1° 45′)	log.	8.484848		
Sin. [$\frac{1}{2}$ [A′ − (B′ − F)] = 8° 30′]	"	9.169702	"	7.654550
Radius (b = 5,483)			"	3.739004

Chord (AF = k_a = 324.8) = 2.a.sin. $\frac{1}{2}$ Xa.

The radius, b, may also be determined by (13), as follows:

$$b = 5{,}473 = \begin{cases} [\frac{1}{2}\, i + a.\text{sin. A}'] \div [+\text{sin. B}' - (\text{cos. F} - \text{cos. I}).a \div i] \\ [25 + 2{,}758] \div [+.9703 - (.9926 - .9848) \times 2{,}865 \div 50] \end{cases}$$

As a difference of 88′ in the value of the radii corresponds to a difference of one minute in the degrees of curvature, the difference of 10′ in the calculated values of b by the two methods is of no importance, and arises from neglecting the fractions of minutes in the computations. When, as in the case under consideration, i is small, and the frog distances nearly equal, i.cot. $\frac{1}{2}$ (I + F) = 334.5 is nearly equal to k_a.

Example 2. (figure 61). A and B, on the outer rail of a 10-degree curve of 5 stations, are the extremities of a chord, (**AB**) = i = 484.9. At these points the two branches of a "Y" turn out from the track by means of 8° 10′ frogs, so that the deflections from the chord **AB** are, 25° to the tangents of the 10-degree curve, and 33° 10′ = A′ = B′ to the frog-tangents. It is required that the radii of the two branches of the "Y" shall be equal, and that they shall intersect at an angle of 8° 10′, so as to connect by a No. 7 frog.

As both of the interior angles, at A and B, of the triangle AIB are acute, by problem V. I must be equal either to (A′ + B′) or to 180° − (A′ + B′); and as the centres are on the same side of **AB**, I is equal to (A′ + B′) = 66° 20′; and as both curves are convex to **AB**, the formulæ are those in table III. corresponding to figure 55½.

As A′ and B′, and the radii a and b, are equal, Xa and Xb are also equal, and may be found by equating the values of Xa

and Xb in the formulæ of table III., or in the following manner:

Problem VI. (1). Xa + F + Xb = I, and therefore 2 Xa = 2 Xb = I−F.

By table III., figure 55½, F=180°−F′=171° 50′, and therefore Xa=Xb=½ (I−F)=½ (66° 20′−171° 50′)=−52° 45′.

Having the value of (Xa=Xb), the radius may be calculated by the formula for a or b, in the table, as follows:

½ (**AB**)=½ i=242.5.....			log.	2.384533
Sin. (A′+½ Xa)=59° 32′.....			"	9.935500
			"	2.320003
Sin. (½ Xb=26° 22′).....log.		9.647600		
Cos. [½ (A′+B′−F)=29° 05′]....	"	9.941469	"	9.589067
Radius (b=538.2), degree of curvature, 10° 40′.....			"	2.730964
Sin. (½ Xa=26° 22′).....			"	9.647600
2.....			"	0.301030
Chord (**AF**) = (**BF**)=478.2=2.r. sin. ½ Xa.....			"	2.679594

The number of stations in each branch will therefore be equal to Xa divided by the degree of curvature, or to 4.96, and from the intersection, F, of the rails, the curves must be those for a switch turn-out for an 8° 10′ frog.

PROBLEM XVII.

CONNECTION OF CURVES BY TANGENT ARCS.

(Figure 64.) Two curves, with radii, a and b, intersect at an angle, F. It is required to determine: (I.) The length, t, of the common tangent, and the positions of its extremities, Ta and Tb, with reference to the intersection, F, of the two curves. (II.) The radius, r, of an arc tangent to the two curves, from a tangent-point on one at a given angular dis-

tance from the extremity, Ta or Tb, of the common tangent. (III.) The points of contact with the two curves, of a tangent arc of given radius, r.

(I.) t is the length of the common tangent; F is the angle between the tangents of the intersecting curves, and Fa and Fb are tangential deflections over the crossing arcs from F to the extremities, Ta and Tb, of the common tangent.

Problem XVI. (18). $\frac{1}{2}\,t = \text{sin. } \frac{1}{2}\,\text{F} \cdot \sqrt{a.b}. \quad \text{cot. } \frac{1}{2}\,\text{F}a = \text{cot. } \frac{1}{2}\,\text{F} + a \div \frac{1}{2}\,t. \quad \text{cot. } \frac{1}{2}\,\text{F}b = \text{cot. } \frac{1}{2}\,\text{F} + b \div \frac{1}{2}\,t.$ (1)

Example. a=1528.2=radius of a 3° 45′ curve. b=996.9= radius of a 5° 45′ curve, and F=19° 07′.

a=1528.2.....	log. 3.184169		
b=996.9.....	" 2.998637		
$a.b$.....	" 6.182806	$\sqrt{a.b}$	log. 3.091403
sin. ($\frac{1}{2}$ F=9° 33′ 30″).....			" 9.220162
$\frac{1}{2}\,t$=204.9.....			" 2.311565

cot. ($\frac{1}{2}$ Fa=4° 16′)=cot. $\frac{1}{2}$ F$+a\div\frac{1}{2}\,t$=13.398.
cot. ($\frac{1}{2}$ Fb=5° 17′)=cot. $\frac{1}{2}$ F$+b\div\frac{1}{2}\,t$=10.805.

(II.) a represents the greater of the two radii, a and b; Ia and Ib are the tangential deflections from the points of contact of the tangent arc with the curves, to the extremities of the common tangent, and I=Ia+Ib is the angle of the tangent arc.

The tangent, t; the radii, a and b, and one of the deflections, Ia or Ib, are given, to find the other, and the radius, r, of the tangent arc.

$$\text{Cot. } \tfrac{1}{2}\,\text{I}b = \text{cot. } \tfrac{1}{2}\,\text{I}a \pm (a-b) \div \tfrac{1}{2}\,t. \quad \text{cot. } \tfrac{1}{2}\,\text{I}a = \text{cot. } \tfrac{1}{2}\,\text{I}b \mp (a-b) \div \tfrac{1}{2}\,t. \quad (2)$$

I=Ia+Ib.

$$r = \begin{cases} a \pm (\frac{1}{2}\,t \div \text{sin. } \frac{1}{2}\,\text{I}) \times (\text{sin. } \frac{1}{2}\,\text{I}b \div \text{sin. } \frac{1}{2}\,\text{I}a). \\ b \pm (\frac{1}{2}\,t \div \text{sin. } \frac{1}{2}\,\text{I}) \div (\text{sin. } \frac{1}{2}\,\text{I}b \div \text{sin. } \frac{1}{2}\,\text{I}a). \end{cases} \quad (2a)$$

For the double sign prefixed to any term substitute the upper sign when the tangent arc is on the outside of the curves, as in figure 64, and the lower sign when it is on the inside, as in figure 64½.

Examples. a = 1528.2; b = 996.9, and t = 409.8.

The tangent curve starts from A, at an angular distance, ($Ia = 49° 40'$), from the extremity, T*a*, of the common tangent.

Required the radius, *r*, of the tangent arc, and the angular distance, I*b*, of its point of contact from the extremity, T*b*, of the tangent.

Cot. ($\frac{1}{2}Ib = 11° 53'$) = cot. $\frac{1}{2}Ia + (a-b) \div \frac{1}{2}t = 2.593 + 2.161 = 4.754$.

$\frac{1}{2}t = 204.9$		log. 2.311565
Sin. ($\frac{1}{2}I = \frac{1}{2}Ia + \frac{1}{2}Ib = 36° 43'$)		" 9.776570
Sin. ($\frac{1}{2}Ib = 11° 53'$)	log. 9.313498	" 2.534995
Sin. ($\frac{1}{2}Ia = 24° 50'$)	" 9.623229	" 9.690269
$x_a = 168.1$		" 2.225264
$x_b = 699.4$		" 2.844726

$r = a + x_a = b + x_b = 1696.3$.

(III.) Given *t*, *a*, and *b*; required the angular distances, I*a* and I*b*, from the extremities of the tangent, to the points of contact of the curves with a tangent arc of given radius, *r*.

r must be either greater or less than both *a* and *b*: greater when the tangent arc is on the outside of the curves, and less when it is on the inside.

$$\left.\begin{array}{l} \text{Sin. } \frac{1}{2}I = \frac{1}{2}t \div \sqrt{(r-a).(r-b)}. \\ \text{Cot. } \frac{1}{2}Ia = \text{cot. } \frac{1}{2}I + (r-a) \div \frac{1}{2}t. \quad \frac{1}{2}Ib = \frac{1}{2}I - \frac{1}{2}Ia. \\ \text{Cot. } \frac{1}{2}Ib = \text{cot. } \frac{1}{2}I + (r-b) \div \frac{1}{2}t. \quad \frac{1}{2}Ia = \frac{1}{2}I - \frac{1}{2}Ib. \end{array}\right\} \quad (3)$$

Example. $a = 1528.2$; $b = 996.9$; $t = 409.8$; required the angular distances, I*a* and I*b*, from the extremities, T*a* and T*b*, of the common tangent, to the points of contact, A and B, of the two curves, with a three-degree, ($r = 1910.1$), tangent arc.

$(r-a) = 381.9$		log. 2.581950
$(r-b) = 913.2$		" 2.960566
$(r-a).(r-b)$		" 5.542516
$\sqrt{(r-a).(r-b)}$. (Deduct from log. of $\frac{1}{2}t$ below).		" 2.771258
$\frac{1}{2}t = 204.91$		" 2.311565
Sin. ($\frac{1}{2}I = 20° 18'$)		" 9.540307

Cot. ($\frac{1}{2}Ib = 7° 57'$) = cot. $\frac{1}{2}I + (r-b) + \frac{1}{2}t = 7.1604$.

$\frac{1}{2}Ia = \frac{1}{2}I - \frac{1}{2}Ib = 12° 21'$.

(IV.) If it is required that the tangent arcs shall start from points, A and B, at given angular distances from Ta and Tb, there will be two branches of the arcs, and the tangent at the point of compound curvature will be parallel to the tangent (**T**a**T**b), as in figure 64¼.

If r_a is the radius of the tangent arc through A, and r_b the radius of the tangent arc through B, the distances, x_a and x_b, by which the radii, a and b, must be increased, are determined by the following formulæ:

$$\left.\begin{aligned} x_a &= [\tfrac{1}{2}t + \sin.(\tfrac{1}{2}\mathrm{I}a + \tfrac{1}{2}\mathrm{I}b)] \times (\sin.\tfrac{1}{2}\mathrm{I}b + \sin.\tfrac{1}{2}\mathrm{I}a). \\ x_b &= [\tfrac{1}{2}t + \sin.(\tfrac{1}{2}\mathrm{I}a + \tfrac{1}{2}\mathrm{I}b)] \div (\sin.\tfrac{1}{2}\mathrm{I}b + \sin.\tfrac{1}{2}\mathrm{I}a). \\ r_a &= a + x_a. \quad r_b = b + x_b. \end{aligned}\right\} \quad (4)$$

In these equations, a is the greater radius; and if (t=o) is a point, a tangent may be inserted by subtracting x_a and x_b, instead of adding.

Example. If $a = 1{,}300$, $b = 1{,}000$, $t = 100$, $\mathrm{I}a = 30°$, and $\mathrm{I}b = 20°$.

$x_a = 79.4$. $x_b = 176.3$. $r_a = a + x_a = 1379.4$. $r_b = b + x_b = 1176.3$.

PROBLEM XVIII.

TURN-OUTS, SIDINGS, AND CROSSINGS FROM RAILROAD-TRACKS.

(*For demonstration see proposition XV.*)

1. *Turn-outs* (figure 59).—**AS** is a radial line across the points, or ends, of a pair of switch-rails on the main track of a railroad.

B is the initial point of the turn-out rail which crosses the main-track rail at a point F. To connect the main and turn-out tracks, the point, S, of the switch-rail, HS, is sprung from S to B.

The distance, (**SB**), is called the throw of the switch, and represented by the letter s. The radial line, **ABS**, is perpendicular to the track-tangent at S, but when the point of the switch moves to B the switch-tangent changes direction by an

angle, S, called the switch-angle, and makes an angle, (B′= 90°−S), with the radial line, **AB**, with which the track-tangent at A makes an angle, (A′=90°).

By the greater portion of the authorities on this subject, as Henck, Trautwine, and others, the switch-rail is supposed to turn, without bending, on the heel at H, and, when the main track is straight, the switch-angle is the angle BHS of the triangle BHS, of which the switch-rail is the base, and the throw the perpendicular, as shown in figure 66¼. When, as in figure 66, the track is on a curve, the switch-rail turns on the heel without change of curvature, compounding with the turn-out curve when the turn-out is to the inside, and reversing when it is to the outside, so that the switch-angle, or change of tangential direction in moving to the switch-tangent, is the same as when the track is straight. In some of the switch formulæ the rail is supposed to bend on a circular curve, which would double the value of the switch-angle based upon the first assumption of a straight rail. In other formulæ, as in those of Searle, the switch-rail is supposed to bend on the curves of the turn-out curves, whatever may be the radii. None of these conditions are realized in practice, but the assumption of a switch-rail bending on a circular curve will be found to be the most correct, as the rail never turns on the heel without bending at all, and it is possible by making the movable portion of the proper length to have it bend, approximately at least, on any required degree of curvature.

There is no need of scrupulous accuracy in switch formulæ, but an error of one degree in the assumed value of the switch-angle may result in an error of two degrees in the degree of the turn-out curve, which is beyond the limits of allowable error. It is for this reason that an experienced trackman will often find it necessary to revise the angles and distances given from the formulæ in ordinary use, in which, although approximate, and correct, at best, only for straight tracks, the degrees of curvature are calculated to the nearest minute, and the frog distances and ordinates to the nearest fraction of the thousandth part of a foot.

In the formulæ of this problem, the angle, B′, between **AB** and the switch-tangent, is equal to 90 degrees minus the switch-angle, whatever it may be. The least value of this angle, in terms of the length, *l*, of the switch-rail, is given by

the following equations, the latter of which is approximate, but sufficiently accurate for all purposes. The maximum value is double the value resulting from the equations.

$$\text{Sin. } S = s \div l. \quad S = 57.s \div l =, \text{ (when } s = .42\text{), to } 24 \div l. \quad (1)$$

Frogs (figure 65).—The crossing of the main track and turn-out rail is made by means of a triangular frame called a frog.

The angle, AFB, between the sides, is the frog-angle, F, and the number, n, of the frog is the ratio of the perpendicular distance, (**FP**) (from point to base), to the width, (**AB**), of the base. The relation between the frog-angle and number is expressed by the following equations, the latter of which is approximate, but sufficiently accurate for all purposes:

$$n = \tfrac{1}{2} \text{ cot. } \tfrac{1}{2} F = 57.2 \div F. \qquad F = 57.2 \div n. \quad (2)$$

For angles corresponding to different frog numbers, see table VIII.

The gauge, g, of the track is the distance between the insides of the rails, and the distance, i, from A to B, is equal to $(g-s)$. AF is the main-track rail, and BF the turn-out rail which crosses it. A is the radius, and Da the degree of curvature, of the main-track rail curve, and b the radius, and Db the degree, of the turn-out rail, FB. The turn-out is to the *inside* when to the concave side of the main track, as for the rails crossing at F_1, figure 59, and to the outside for the curves crossing at F_2 and F_3. The main-track rail, AF, is convex to **AB** when the turn-out is to the inside, and concave when it is to the outside. The turn-out rail, BF, is concave to **AB** when the turn-out is to the inside, and may be either concave or convex when the turn-out is to the outside, as in the case of the rail-curves BF_2 and BF_3.

The chords of the arcs, AF and BF, are called the frog-distances, and represented by the letters k_a and k_b. For convenience of reference the symbols described above will be recapitulated:

AF is the curve of the main-track rail, a the radius, Da the degree, and (**AF**)$=k_a$ the chord, or frog, distance from A. BF is the curve of the turn-out rail cutting AF at F; b is the radius, Db the degree, and (**BF**)$=k_b$ the chord, or frog, distance from B.

(**AB**) = ($g-s$) is the distance from A to B, and F and S are the frog and switch angles. i, S, F, and Da are generally given, to determine the frog-distances and the degree, Db, of the turn-out curve.

2. *Solution by Diagram.*—The position of the frog, when the track is located, may be determined by the following method: From the middle of (**AB**) measure, parallel to the track-tangent, the distance, $o=\frac{1}{2}(g-s)$.cot. $\frac{1}{2}$(F+S), to a point, O. From O as a centre, with a radius, r, equal to $\frac{1}{2}(g-s) \div$ sin. $\frac{1}{2}$(F+S), describe a circle cutting the main-track rail at a point F. This will be the position of the frog, whatever may be the curvature of the track, or whether the turn-out is to the inside or outside. The same method is applicable in the case of double switches, sidings, and crossings, substituting for ($g-s$) and (F+S), ($c-g$) and F in the case of sidings; ($c-2g$) and Fa+Fb in the case of crossings; and ($g-2s$) and (F+2 S) in the case of double or three-throw switches, as more particularly explained under those heads.

Construction of Diagram for Turn-outs. (Plate 14).—The diagram may be commenced thirty or forty feet from the end, or toe, of the switch-rail, as there will be no intersections in that distance for a single switch on a standard gauge. A section only of the diagram is shown in the figure, commencing 50 feet from the toe of the switch. **AI** is the main-track or tangent rail when the track is straight. The curves from **AI,** as a tangent, *above* and *below*, are the rails of the main track for turn-outs to the *outside* and *inside*, the lower curves corresponding to turn-outs to the inside. These curves are platted by tangential ordinates from table 1. The radii are equal to the radii of the centre lines of the track, plus $\frac{1}{2}g$ when the turn-out is to the outside, and minus $\frac{1}{2}g$ when it is to the inside.

They may, without appreciable error, be assumed equal to the radii of the centre lines.

The circles cutting these rail-curves are frog-circles of degrees, or angles, marked at their upper extremities. The centres of these circles are on the line **MO,** parallel to **AI,** at a distance equal to $\frac{1}{2}(g-s)$, which is equal to 2.145 for a gauge of four feet eight and a half inches and a throw of five inches.

The scale on the line **AI** is platted from A as the initial

point, and from this scale the distances may also be measured on the line **MO**, as M and A are both on the radial line **AB** across the ends of the switch-rails. The distance, o, from M to the centre of a frog-circle of which O is the angle, is equal to $\frac{1}{2}(g-s)$.cot. $\frac{1}{2}$O, and the radius of the circle is $[r=\frac{1}{2}(g-s)\div$ sin. $\frac{1}{2}$O]. o is equal to half the frog-distance (**AF**), and r to half the frog-distance (**BF**), given in tables IV., V., VI., VII., and VIII., for values of F corresponding to O, in the case of single and double switches, for standard and narrow gauges, and sidings with 10 feet between the intersecting rails. As each circle cuts the line **MO** at a distance, $o+r$, from **M**, they may be platted by laying off that distance from the tables, and platting the circles by means of the following table of tangential ordinates from a tangent perpendicular to **MO**.

r	n	Ordinates at proportional parts of n.					
		$\frac{1}{4}n$	$\frac{1}{2}n$	$\frac{3}{4}n$	n	$1\frac{1}{4}n$	$1\frac{1}{2}n$
50	10	.06	.25	.57	1.01	1.59	2.30
40	8	.05	.20	.46	.81	1.27	1.84
30	6	.04	.15	.34	.61	.96	1.38
20	4	.02	.10	.23	.40	.64	.92
10	2	.01	.05	.12	.20	.32	.46

To plat any circle, as that with radius 50, draw a perpendicular to **MO**, at the point at which it is cut by the circle, and at one-fourth, one-half, three-fourths, etc., of the distance $n=10'$, draw the ordinates to the circle given in the table, for those proportional parts of n. For radii intermediate to those given in the table the ordinates may be calculated by simple proportion. The ordinate, e, for abscissas, n, small in proportion to the radius, may be calculated by the formula, $e = .51 \cdot n^2 \div r$.

The ordinate at the distance n is the middle ordinate of a chord of length $2n$.

If F and S are the frog and switch angles, the position of the frog is found by following the frog-circle of angle, (F + S), to the intersection with the rail-curve, above **AI** when the turn-

out is to the outside, and below **AI** when the turn-out is to the inside.

If, for example, F = 7° 00′, S = 1° 20′, Da = 8° 22′, and the turn-out is to the inside, the frog will be at the intersection of the (F + S)=8° 20′ frog-circle, with the 8° 22′ rail-curve below **AI**, at a distance of 59.05 from A by scale. When the frog is at a perpendicular distance, (**FP**) = p, above or below the tangent **AI**, read the tangent distance (**AP**) from the scale on **AI**, and add, for the distance on the hypothenuse, (**AF**), the correction, c, corresponding to (**AP**) and (**PF**) in the following table:

Table P, of corrections, c, to be added to the base (**AP**) *to obtain the hypothenuse* (AF) = k_a, *when the perpendicular* (**FP**) = p *is given.*

(PF)=p	(**AP**)	30	40	50	60	70	80	90	100
1	c	.02	.02	.01	.01	.01	.01	.01	.00
2	c		.05	.04	.03	.03	.03	.02	.02
3	c			.09	.08	.07	.06	.05	.04
4	c			.16	.14	.12	.10	.09	.08
5	c				.22	.18	.15	.13	.12
6	c				.31	.26	.22	.20	.18
7	c					.35	.30	.27	.25
8	c					.45	.40	.35	.32
9	c						.50	.45	.41
10	c						.60	.56	.51
11	c							.67	.60
12	c							.80	.70

If, for instance, the tangent distance is 90 feet, and the frog is 9 feet above or below **AI**, the distance (**AF**) would be 90.45. As a rule, it will be sufficient to neglect the correction, and take the distance measured along the tangent **AI**, to the foot of a perpendicular from F.

The switch-tangent through B makes an angle S with the

line **AI**, and, by platting the turn-out curves from that tangent, their degrees might be read from the diagram for any intersection, but this would necessitate the construction of a diagram for each different value of the switch-angle, and the degree of the turn-out may be obtained by approximate formulæ which are sufficiently exact for all purposes. With a straight main track, and a switch-angle equal to zero, and a frog-angle F, the frog would be at the intersection of the F-degree frog-circle with the line **AI**, and the degree, Do, of the turn-out curve through that point would be determined by the formula.

$$\left.\begin{array}{ll} \text{Prop. XV. (5).} & \text{Sin. } \tfrac{1}{2}Do = 100.\text{vers. } F \div 2.(g-s). \\ \text{“ “ (40).} & Do = .874.F^2 \div (g-s), \text{ (nearly).} \end{array}\right\} \quad (3)$$

The values of Do for different values of F, at intervals of ten minutes, are given in the frog tables, and marked on the diagram at each intersection of the line **AI** with a frog-circle.

With a straight main track, and a frog and switch angle, F and S, the frog would be on the line **AI**, at its intersection with the F+S frog-circle, and the degree of the turn-out curve would be determined by the following formulæ, the latter of which is approximate, but sufficiently exact for all cases occurring in practice:

$$\left.\begin{array}{ll} \text{Prop. XV. (5).} & \text{Sin. } \tfrac{1}{2}Ds = 100.(\cos. S - \cos. F) \div 2(g-s). \\ \text{“ “ (10).} & Ds = .874.(F^2 - S^2) \div (g-s). \end{array}\right\} \quad (4)$$

In the above formulæ, Ds represents the degree of the turn-out curve for a straight track and a switch-angle S, and Do the degree when S=0. By (40), Ds=Do−.874 . $S^2 \div (g-s)$.

With a gauge $g = 4.708$, and a throw of 5 inches, $(g-s) = 4.29$, and the correction will be given by the following formulæ:

$$Ds = Do - .204.S^2 = Do - 12.5 \, . \, S^2 = Do - C. \quad (4\tfrac{1}{2})$$

In these equations, S is to be expressed in degrees and decimals, the first equation, $.204.S^2$, giving C in degrees, and the second, $12.5.S^2$, in minutes. The correction, C, is given in the following table for different values of the switch-angle:

Table S, for determining the value of C *in the equation,* $Ds = Do - C$.

S	C
0° 0′	0° 00′
0° 50′	0° 09′
1° 0′	0° 12′
1° 10′	0° 17′
1° 20′	0° 22′
1° 30′	0° 28′
1° 40′	0° 35′
1° 50′	0° 42′
2° 0′	0° 50′
2° 10′	0° 59′
2° 20′	1° 08′
2° 30′	1° 18′
2° 40′	1° 29′
2° 50′	1° 41′
3° 0′	1° 53′
3° 10′	2° 06′
3° 20′	2° 19′
3° 30′	2° 33′
3° 40′	2° 48′
3° 50′	3° 03′
4° 0′	3° 20′
4° 10′	3° 37′

From the value of Ds thus determined, the degree, Db, of the turn-out curve may be obtained when the degree, Da, of the main-track rail is given, by the following formula

$$Db = Ds \pm Da = Do - 12.5.S^2 \pm Da = Do - C \pm Da. \quad (5)$$

In this equation, Do is the degree at the intersection of the F-degree frog-circle with the line **AI**, and C is the correction from table S, above, for the given value of S.

For the double sign prefixed to Da substitute a *plus* sign when the turn-ou is to the *inside*, and a *minus* sign when it is to the *outside*. For a turn-out to the outside, the curve will be concave to **AB** when Da is less than Ds, and convex when it is greater, as shown in the curves crossing at F_2 and F_3, figure 59.

Examples of Use of Table.—Given the frog and switch angles, F *and* S, *and the degree,* Da, *of the main-track rail; required the frog-distance* (AF). The frog will be at the intersection of the (F + S) frog-circle with the Da rail-curve, above **AI** when the turn-out is to the outside, and below when it is to the inside. If F = 7° 00′, S = 1° 20′, Da = 8° 22′, the frog is at the intersection of the (F + S 8° 20′) frog-circle with the 8° 22′ rail-curve above **AI**, for a turn-out to the outside. It is not necessary to consider the turn-out to the inside, because the degree, Db, of the turn-out would, in such case, be equal to Do − C + Da = 17° 58′, which is a greater curvature than should be used. It is for this reason that the rail-curves

below **AI**, at intersection with the 8° 10′ frog-circle, extend only to 7 degrees.

For a turn-out to the outside, the distance (**AP**) along **AI**, to the foot of a perpendicular from F, is 59.0. The frog is, however, 2.5 feet above **AI**, and therefore (**AF**) is the hypothenuse of a right-angled triangle of which 59.0 is the base and 2.5 is the perpendicular, and (**AF**) is therefore equal to 59.0, plus the correction, .04, in table P, for (**AP**) = 59.0, and p = 2.5. It is to be observed that the distance (**AP**) is not the distance to the intersection of **AI** by the frog-circle, but to the foot, P, of a perpendicular from F to **AI**.

As a rule, (**AF**) may be assumed equal to (**AP**).

Example 2. *Given* F, S, *and* Da; *required* Db.

Formula. $Db = Do - C \pm Da$.

In this equation, Do is the degree at the intersection of the F-degree frog-circle with the line **AI**; C is the correction, from table S, for the given switch-angle, and the double sign will be plus when the turn-out is to the inside, and minus when it is to the outside. If, as in the preceding example, F=7° 00′; S =1° 20′, and Da=8° 22′; Do, on the (F=7° 00′) frog-circle, is 9° 58′; the correction, C, from table S, for (S=1° 20′) is 22′; and, therefore, Db is equal to 9° 58′ − 0° 22′ − 8° 22′ = 1° 14′ for an outside turn-out, and to 9° 58′ − 0° 22′ + 8° 22′ = 17° 58′ when the turn-out is to the inside. With Da=12° 36′, Db would be −3° 00′ for a turn-out to the outside, the negative sign indicating that the turn-out is convex to **AB**, as in the case of the arc BF_3 in figure 59.

Example 3. *Given* S, Da, *and* Db; *required* F. F will be equal to the degree of the frog-circle cutting **AI** at the point where Do is equal to D$b \pm$ Da+C. In this equation, C is the correction, from table S, for the given value of S, and the sign prefixed to Da will be plus for an outside, and minus for an inside, turn-out. If, for instance, S = 1° 20′, Da = 8° 22′, Db =1° 14′, and the turn-out is to the outside, Da+Db+C=8° 22′ +1° 14′+0° 22′= 9° 58′; and as this is the angle, Do, at the intersection of the 7° 00′ frog-circle with **AI**. F is equal to 7° 00′. With Db equal to 17° 58′, and the turn-out to the inside, Db−Da+C=17° 58′−8° 22′+0° 22′=9° 58′, and the frog-angle would therefore be 7° 00′, as for the turn-out to the outside.

These examples are taken from Henck's formulæ; the pro-

cess for the frog-distance is mathematically correct, and the values by diagram are correct within one minute for F, and within three minutes for D*b*. The same process is applicable to sidings and crossings, for each of which the line **AB** is supposed to be at right angles to the track-tangent at A. In the case of a three-throw switch, both of the tangents at A and B are switch-tangents, and it is therefore necessary to make a diagram for each value of the switch-angle.

In this case, $o''=(g-2.s).\cot.\frac{1}{2}(F''+2S)$, and $r''=(g-2.s)\div\sin.\frac{1}{2}(F''+2S)$, and o'' is equal to half the chord distance, (**AF**), and r'' to half the chord distance, (**BF**), given in the tables of problem XVIII. for three-throw switches. By constructing the diagrams for single and double switches on the same sheet, all frog problems could be readily solved. This could be easily done, as the frog-circles for the double switch would be within the frog-circles of the single switch, and they would not cross or interfere. The turn-out and main-track curves might be distinguished by different colors.

For convenience of scaling, the diagram should be made on cross-section paper, and on a scale not less than one-quarter of an inch to the foot.

3. *Formulæ for Turn-outs.*—Exact formulæ for any curvature of the main track are given in proposition XV., but the problems may be solved with sufficient accuracy by assuming that the frog-distances are the same for curved and straight main tracks. The degrees of curvature may be determined by approximate formulæ based on (40), proposition XV. In tables IV. and VI., for standard and narrow gauges, the frog distances are given for straight tracks, and also the formulæ for determining the degrees of the turn-out curves. Table IX., for sidings, crossings, and single and double turn-outs, for standard and narrow gauge tracks, is based on (39) and (40), proposition XV. For explanation of formulæ for chord and curve ordinates, see problem XIV.

The use of these tables will be understood by the following examples, in all of which the gauge is 4.708 and the throw 5 inches.

Example 1. *Given* F *and* S; *required the frog-distances,* (**AF**) *and* (**BF**), *the middle and side chord ordinates of the turn-out curve, and the middle and side curve ordinates between the two rails.*

The frog-distances are given in tables IV. and VI., opposite

the angle (F+S). They may also be calculated by the formula, $k = 491 \div (F+S)$, in table IX. The curve and chord ordinates are calculated by table IX., as in the succeeding example.

Required, for example, the frog-distances and ordinates when F=7° 00′ and S=1° 20′.

Table IV., opposite 8° 20′, (**AF**)=58.9. (**BF**)=59.1.

Table IX. (**AF**)=491÷(F+S)=58.9. (**BF**)=(**AF**)+0.2=59.1.

The turn-out curve may be laid down, without calculation of the degree of curvature, by middle and side ordinates from the chord (**BF**), when the degree, Da, of the main track is given, or by curve ordinates from the main-track rail, which are the same for all degrees of curvature.

Chord Ordinates. The side ordinates, or ordinates from the chord at one-fourth and three-fourths of its length, may be assumed equal to three-fourths of the middle ordinate, $o\frac{1}{2}$, calculated as follows:

When the main track is straight, so that Da *is equal to zero:*

Prob. XIV. (5). $o\frac{1}{2} = .218 \,.\, (F-S) \,.\, k \div 100 = .218 \times 5.67 \times .59 = 0.73.$

" (7). $o\frac{1}{2} = \frac{1}{4}(g-s).(F-S) \div (F+S) = 1.07 \times 5.67 \div 8.33 = 0.73.$

When Da *is the degree of curvature of the main track:*

Let $o\frac{1}{2}$ be the middle ordinate of the turn-out curve for a straight main track, and $o'\frac{1}{2}$ for the curved track.

Prob. XIV. (7a). $o'\frac{1}{2} = o\frac{1}{2} \pm .218 \,.\, Da \,.\, (k \div 100)^2.$

" " $o'\frac{1}{2} = .218 \,.\, k \div 100 \,.\, [(F-S) \pm (k \div 100) \,.\, Da].$

In this equation, problem XIV. (5), $.218 \,.\, Da \,.\, (k \div 100)^2$, is the middle ordinate for the curve AF.

For the double sign in this equation substitute a plus sign for an inside, and a minus sign for an outside, turn-out.

If Da=8° 22′=8.37°, F=7° 00′, S=1° 20′=1.33°, the chord, k, as calculated above, is 59, and the middle ordinate, $o\frac{1}{2}$, for a straight main track, is .73.

$o'\frac{1}{2} = o\frac{1}{2} + .218 \times 8.37 \,.\, (.59)^2 = .73 + .63 = 1.36$ for an inside turn-out.

$o'\frac{1}{2} = o\frac{1}{2} - .218 \times 8.37 \,.\, (.59)^2 = .73 - .63 = 0.10$ for an outside turn-out.

The calculation by the second equation of (7) is as follows:

$o'1/2 = .218 \times .59 . (5.67 + .59 \times 8.37) = 1.37$ for an inside turn-out.
$o'1/2 = .218 \times .59 . (5.67 - .59 \times 8.37) = 0.10$ for an outside turn-out.

To bend a rail of length, l, the middle ordinate $= o'1/2 \times (l \div k)^2$. Side ordinates $= 3/4$ middle ordinate.

Curve ordinates are the distances between the intersecting rails, measured on the radial lines of the main-track rail, at one-fourth, one-half, and three-fourths of the chord distance $(\mathbf{AF}) = k$.

These ordinates, represented by $q1/4$, $q1/2$, and $q3/4$, are independent of the degrees of curvature of the intersecting rails, and constant for the same values of i and S, and are determined by problem XIV. (9), or by table IX. in the following manner:

If S is equal to zero, the ordinates are $\frac{1}{16}$, $\frac{12}{16}$, and $\frac{7}{16}$ of $i = (g - s)$, and are given in line 5, table IX. For any value of S multiply the corresponding numbers in table C by $k \div 100$, and subtract the product from the ordinates in line 5.

When $i = 4.29$, the ordinates in line 5 are 4.02, 3.22, and 1.88.

The numbers in table C for S = 1° 20′ are .44, .58, and .44, and these multiplied by .59 and subtracted from 4.02, 3.22, and 1.88, give 3.75, 2.86, and 1.61 for the curve ordinates.

Should the degree, Db, of the turn-out curve be required, it may be determined by the formulæ of the next example.

Example 2. *Given* F, S, *and the degree,* Da, *of the main-track rail; required the degree,* Db, *of the turn-out rail.*

Table VI. $Db = Do - C \pm Da$.

Table IX. $Db = \begin{cases} .204\,(F^2 - S^2) \pm Da. \\ 100\,.(F - S) \div k \pm Da. \end{cases}$

In table IV., Do is the angle opposite F, and C the correction for S in table C. In the formulæ of both tables, Da is to be plus when the turn-out is to the inside, and minus when it is to the outside. If, for an outside turn-out, Da is greater than $Do - C$, or than $.204\,(F^2 - S^2)$, the turn-out is convex to **AB**, as in the case of the curve BF_3 in figure 59.

If, for example, F = 7° 00′, S = 1° 20′, Da = 8° 22′, the angle, Do, opposite to (F = 7° 00′) in table IV., is 9° 58′, and the cor-

rection, C, for S=1° 20′, is 0.22′, and, therefore, Db is equal to 9° 58′−0.22′−8° 22′=1° 14′ when the turn-out is to the outside, and to 9° 58′−0.22′+ 8° 22′= 17° 58′ when it is to the inside.

By table IX. Db is equal to .204.(F−S).(F+S)−Da=1° 16′ when the turn-out is to the outside, and to .204.(F−S).(F+S) +Da = 18° 00′ when it is to the inside. The exact values of Db, calculated by (9), proposition XIV., are 1° 15′ for the outside turn-out, and 18° 00′ for the inside. From the chord, k, and the degree, Db, thus calculated, the middle chord ordinate, $o\frac{1}{2}$, and the side ordinates, $o\frac{1}{4}$ and $o\frac{3}{4}$, may be determined in the following manner:

Table 1. $o\frac{1}{2}$ = tangential ordinate for an abscissa equal to $\frac{1}{2}k$.

Problem XIV. (5). $o\frac{1}{2}$=.22.Db.(k÷100)².

The chord, k, in this instance is 59, and the ordinates as follows:

Db=1° 14′. Table I. $o\frac{1}{2}$=tangential ordinate for 29.5 = .09.
Db=18° 00′. $o\frac{1}{2}$=tangential ordinate for 29.5 =1.38.
Db=1° 14′=1.23°. Prob. XIV. (5). $o\frac{1}{2}$=.22× 1.23.(.59)²= .09.
Db=18° 00′. $o\frac{1}{2}$=.22×.18×(.59)²=1.38.

The side ordinates may be assumed equal to three-fourths of the middle ordinates.

Example 3. *Given* S, Da, *and* Db; *required* F.—Table IV. F is the frog-angle opposite the angle Do equal to Db±Da+C. In this equation, C is the correction, from table S, for the given value of S, and the sign prefixed to Da will be plus for an outside, and minus for an inside, turn-out. Table IX. F²=4.91.(Db±Da)+S². In this equation, also, the sign of Da will be plus for an outside, and minus for an inside, turn-out. Da = 8° 22′, Db = 1° 14′, S = 1° 20′, the turn-out to outside. In table IV. the angle, (Do=1° 14′+8° 22′+0° 22′=9° 58′), is the angle opposite to the angle (F=7° 00′) in the left-hand column, and this is, therefore, the value of F.

Table IX. $F=\sqrt{4.91\,.\,(1°\ 14' + 8°\ 22')} + S^2 = \sqrt{48.91} = 7°\ 00'$.

4. *Three-throw Switches* (figure 68).—The formulæ for turnouts from three-throw switches are given in proposition XV., but the problems may be solved with sufficient accuracy by means of tables V. and VII., problem XVIII., or by table IX.

If one of the turn-out rails is laid down, the position of a middle frog, of given angle F″, on that rail, may be determined in the following manner: From the centre of the track, and parallel with the track-tangent, measure to a point, O″, the distance, $o'' = \frac{1}{2}(g - 2s).\cot.\ \frac{1}{2}(F'' + 2S)$; and from O″ as a centre, with a radius, $r'' = \frac{1}{2}(g - 2s) \div \sin.\ \frac{1}{2}(F'' + 2S)$, describe a circle cutting the located turn-out rail at a point F″. F″ will be the position of the middle frog on the located rail. The frog-distance on the main track may be determined as in example 1 of the last article, and the turn-out curve laid down by curve ordinates from the main-track rail by the formulæ of table IX. The application of the tables to three-throw switches will be understood by the following example, for a gauge of 4.708 and a throw of five inches.

Example 1. *Given the main-track frog-angles,* F*a and* F*b, and the switch-angle* S; *required the middle frog-angle,* F″, *and the frog-distance,* k''.

By Table V. The degrees, D*b* and D*a*, of the turn-out curves must be determined from table IV., as explained in example 2 of the preceding article. F″, in table V., will then be the frog-angle opposite to the angle $Do'' = Da + Db + C''$, C″ being the correction from table S″ for the switch-angle, S. The frog-distances, k_a'' and k_b'', are the distances opposite to the angle $(F'' + 2S)$.

$$\text{By table IX.}\quad F^{2''} = .9\,.\,(F^2a + F^2b) + 2.2 \times S^2. \quad (1)$$

$$k'' = 443 \div (F'' + 2S). \quad (2)$$

F″ may also be determined in terms of D*b* and D*a* as follows:

$$F^{2''} = 4.43\,(Db + Da) + 4 \times S^2. \quad (3)$$

If one of the turn-out curves is given, the other may be determined by the following equation:

$$(Da + Db) = .226\ F^{2''} - .9 \times S^2. \quad (4)$$

In these and other equations, in which the angles enter as multiples or divisors, the minutes should be reduced to decimals of a degree by the rules of problem I. The squares may be taken from a table of squares, or the difference of the squares calculated as the multiple of the sum and difference of the angles.

Example 1. The degree, D, of the main track curve is 5° 00′; S=1° 00′. The main-track frog-angles are, Fa=6° 00′ on the inside, and Fb = 9° 00′ on the outside. Required the middle frog-angle, F″, the middle frog-distances, (**AF**″) and (**BF**″), and the curve ordinates from the main-track to the turn-out curves.

Table IV. Fa=6° 00′. Da=Do+D−C=7° 19′+5° 00′− 0° 12′
=12° 07′.
Fb =9° 00′. Db=Do+D−C=16° 30′−5° 00′−0° 12′
=11° 18′.

Table V. (F″=10° 21′)= angle opposite (Do″=12° 07′+11° 18′
+0° 55′=24° 10′).

In this equation, 0° 55′ is the correction for Do″ for a switch-angle of 1° 00′, and Da and Db are the turn-out curves calculated above. The middle frog-distances are the distances, (**AF**″)=35.9, and (**BF**″)=36.1, opposite the angle (F″+ 2 S)= (12° 21′) in table V.

Table IX. $F'' = \sqrt{.9 . (F^2a + F^2b) + 2.2 \times S^2} = \sqrt{.9 \times (117.) + 2.2}$
=10° 21′.
$k'' = 443 \div (F'' + 2S) = 443 \div 12.35 = 36.1.$

The sum, (Da+Db), of the degrees of the turn-out curves may also be calculated as follows by (4):

$$(Da + Db) = .226 \times F^{2\prime\prime} - .9 \times S^2 = 23.40° = 23° 24'.$$

The exact value of the middle frog-angle calculated by (16), proposition XIV., is 10° 23′.

The turn-out curves may be laid down by the curve ordinates from the main-track rails which they intersect, by table IX., as explained in example 1 of the preceding article, and their intersection will give the position of the middle frog. The position may also be determined in the following manner: F and k are the main-track frog angle and distance; F″ and k'' the middle frog angle and distance; $p = k'' \div k$; and q the ordinate from the main track rail to F″.

q may be determined by the following formula deduced from (8), problem XIV.:

$$q = (g - s) . (1 - p) . [(1 + p) . F + (1 - p) . S] \div (F + S). \quad (5)$$

In the case under consideration, S=1° 00′; Fa=6° 00′; Fb =9° 00′; k''=36.0; k_a= 70.2; k_b= 49.0; $k''\div k_a$=.51; $k''\div k_b$= .73.

p=.51. q=(4.29).(.49).[(1.51).(6)+(.49)] ÷ 7=2.87.
p=.73. q=(4.29).(.27).[(1.73).(9)+(.27)]÷10=1.81.

As these distances are measured from the opposite rails of the main track to the middle frog, the sum should be equal to the gauge, 4.71, from which it differs only by .03.

5. *Sidings* (figure 69).—In sidings, the siding-rail is produced from the frog-tangent at F until parallel to the main track at B, and as **AB** is a radial line to both curves, A′=B′=90°, and I = S = 0. If c is the distance between the track-centres, the distance from A to B is equal to $(c-g)$, and the formulæ are those for a turn-out, except that $(c-g)$ is to be substituted for $(g-s)$, and S is equal to zero, and the quantities may therefore be taken from tables IV. and VI. by multiplying the distances, and dividing the angles, by $(c-g)\div(g-s)$. The exact formulæ are given in proposition XV., and table VIII. is calculated for a distance (**AB**) = 10, but the problems may be solved with sufficient accuracy by the formulæ of table IX., which are calculated for distances, between track-centres, of 12, 14, and 16 feet. The examples are taken from values calculated for a distance of 14 feet between track-centres, or for a distance, (**AB**) =9.29.

The siding will be on the inside when on the concave side of the main track, as for the curve BF, figure 50¾, and on the outside when on the convex side of the main track, as in the case of the curves BF1½ and BF1¼. In the formulæ, k_a and Da will be the frog distance and degree of curvature of the main-track rail, and k_b and Db of the siding-rail BF.

As S is equal to zero, the calculations may be facilitated by using the frog-number, which may be calculated by the formula, n=57.2÷F.

The position of the frog may be determined in the following manner: From the middle of (**AB**) to a point, o, measure parallel to the track-tangent the distance, $o = \frac{1}{2}(c-g)$.cot. ½ F, and from O as a centre, with a radius, $r = \frac{1}{2}(c-g)\div$ sin. ½ F, describe an arc cutting the main-track rail at a point F. F will be the point of the frog. The distances, o and r, may be assumed equal to $(c-g).57.2\div$F, or to $(c-g).n$.

On the same principle a diagram may be prepared, as explained in article 2 with reference to turn-outs. A diagram for turn-outs may be used for sidings by multiplying the frog-distances, and dividing the degrees, D*o*, of the turn-out curves, by $(c-g)\div(g-s)$.

Example 1. *Given the frog-angle,* F, *and the distance,* (**AB**)$=(c-g)$; *required the frog-distances, and the curve ordinates from main to turn-out rail, at one-fourth, one-half, and three-fourths of the frog-distance.*

Table VIII. Multiply the frog-distances in the table corresponding to F by one-tenth of $(c-g)$.

Table IX. $\left.\begin{array}{l}(\mathbf{AF})=2\,(c-g).n=(c-g).114.4\div\mathrm{F}.\\(\mathbf{BF})=(\mathbf{AF})+4\,(c-g)+100.\end{array}\right\}$ (1)

The curve ordinates at $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$ of the frog-distance from A will be equal to $\frac{5}{16}$, $\frac{8}{16}$, and $\frac{7}{16}$ of $(c-g)$. They should be measured at right angles to the track-tangent from A, or, nearly enough, on the radial line of the arc AF*a*.

The distance between track-centres is equal to 14 feet, the distance (**AB**) to 9.29, and the frog-number, *n*, to 7, corresponding to a frog-angle, F, of 8° 10′.

Table VIII. $(\mathbf{AF})=140.0\times.93=130.2.$
$(\mathbf{BF})=140.4\times.93=130.6.$
Table IX. $(\mathbf{AF})=2\,(c-g)\,.\,n=130.2=10.63+8.17=130.1.$

The curve ordinates, as given in the table, are 8.71, 6.96, and 4.06.

The chord ordinates are determined as for turn-outs, making $S=o$, and substituting $(c-g)$ for $(g-s)$.

Example 2. *Given* F, $(c-g)$, *and* D*a*; *required* D*b*.

Table VIII. $Db=10\,.\,[Do\div(c-g)]\pm Da.$ (2)
Table IX. $Db=.874\,\mathrm{F}^2\pm Da=100\times\mathrm{F}\div k\pm Da.$ (3)

In (2), D*o* is the angle in the table opposite the angle F; and in (2) and (3), the sign prefixed to D*a* will be plus for an inside, and minus for an outside, siding. When the siding is to the outside, BF will be concave to **AB** for positive values of D*b*, and convex for negative values, as in the cases of the arcs BF_2 and BF_3 in figure 50.

$$Da=5°\ 00'.\quad F=8°\ 48'=8.8°.\quad (c-g)=9.29.$$

(2) Table VIII. $Db=10\times 6.75\div 9.29+5°=12°\ 16'$ for inside siding.

$Db=10\times 6.75\div 9.29-5°=2°\ 16'$ for outside siding.

(3) Table IX. $Db=.094\times F^2+5°=12°\ 17'$ for inside siding.

" " "$-5°=2°\ 17'$ " outside "

In (2), $Do=6°\ 45'=6.75°$ is the angle opposite to ($F=8°\ 48'$). The exact value of Db, from (27), proposition XV., is $12°\ 15'$.

Example 3. *Given* $(c-g)$ *and the frog-distance,* k*; required the frog-angle* F.

$$F=(c-g).(114.4)\div k.\quad n=k\div 2.(c-g).\quad (4)$$

If $(c-g)=9.29$, and $k=130.2$, $F=114.4\,.\,(9.29)\div 130.2=8.17°=8°\ 10'$. $n=130.2\div 18.58=7.00$.

6. *Crossings* (figure 70).—Fa is the greater and Fb the lesser frog-angle. B is at the distance g from the frog Fb, measured at right angles to the frog-tangent, and **AB** is on a radial line of the arc AFa, and therefore $A'=90°$, and $B'=90°-Fb$.

The distance (**AB**) is equal to $c-g\,.\,(1+\cos.\ Fb)$, which may be assumed equal to $(c-2g)$, c being the distance between the centres of the parallel tracks, and g the gauge.

In this case the rails intersect at Fa; **AB** makes an angle $A'=90°$ with the tangent at A, and an angle, $90°-Fb=B'$, with the frog-tangent at Fb, and therefore the formulæ are the same as for turn-outs when $(g-s)=(c-2g)$, and $S=Fb$.

The position of one frog, as Fa, may be obtained by measuring from the middle of (**AB**) to a point O, on a parallel to the track-tangent at A, the distance $o=\frac{1}{2}(c-2g)\,.\,\cot.\ \frac{1}{2}(Fa+Fb)$, and from O as a centre, with a radius, $r=\frac{1}{2}(c-2g)\div\sin.\ \frac{1}{2}(Fa+Fb)$, describing an arc cutting the rail through A at a point Fa. Fa will be the point of the other frog.

The distances o and r may be taken from table IV. by multiplying the chords (**AF**) and (**BF**), corresponding in that table to $(F=Fa+Fb)$, by $(c-2g)\div(g-s)$.

In practice, o and r may be assumed equal to $(c-2g)$ multiplied by 57.2 and divided by $(Fa+Fb)$.

The exact formulæ are given in article 4, proposition XV., but the examples are solved by means of tables IV. and IX.

Example 1. *Given* Fa, Fb, *and the distance* (**AB**) = ($c-2g$); *required the frog-distance*, (**AF**a), *and the curve ordinates between the rails* AFa *and* BFa.

Table IV. The frog-distance, (**AF**a), is equal to the frog-distance for an angle (F=Fa+Fb), multiplied by ($c-2g$) ÷ ($g-s$).

Table IX. (**AF**a) = k = ($c-2g$) . (114.4) ÷ (Fa + Fb). (1)

The curve ordinates are obtained by multiplying the corrections in table S, for Fb = S, by $k \div 100$, and subtracting the product from the ordinates in line 5, which are equal to $\frac{5}{16}$, $\frac{8}{16}$, and $\frac{7}{16}$ of ($c-2g$). The chord ordinates are determined as for turn-outs, substituting ($c-2g$) for ($g-s$), and Fb for S.

Fb = 6° 00′; Fa = 8° 00′; distance, c, between parallel track-centres fourteen feet; (**AB**) = ($c-2g$) = 4.58.

Table IV. The frog-distance for (F=Fa+Fb=14°) in table IV. is 35, which, multiplied by (4.58) ÷ (4.29), gives (**AF**a) = 37.4.

Table IX. (**AF**a) = 524 ÷ 14 = 37.4.

If Fb were zero, the curve ordinates would be those given in line 5 of the table, equal to 4.29, 3.43, and 2.00.

From these subtract the corrections multiplied by .37, in table S, corresponding to S=Fb=6° 00′, giving for the curve ordinates, at one-fourth, one-half, and three-fourths of (**AF**a), 3.57, 1.46, and 1.28.

Example 2. *Given* ($c-2g$), Fa, Fb, *and the degree*, Da, *of the rail* AFa; *required the degree*, Db, *of the rail-curve* BFa.

Table IV. Db = Da + (Do — .204×Fb^2).($g-s$) ÷ ($c-2g$). (2)

Table IX. Db = Da + .874 (Fa^2 — Fb^2) ÷ ($c-2g$) = Da + 100 × (Fa — Fb) + k. (3)

In (2), Do is the angle, in column Do, corresponding to (F = Fa + Fb).

Let ($c-2g$)=4.58, Fa=8° 00′, Fb=6° 00′, and Da=6° 00′.

Table IV. Db = 6° 00′ + (13° 02′ — .204 × Fb^2) × (4.29) ÷ (4.58) = 11° 20′.

Table IX. Db = { 6° 00′ + .19 . (64—36) = 11.32° = 11° 19′. / 6° 00′ + 100 . (8° — 6°) + 37.4 = 11.34° = 11° 20′. }

When Fa=Fb, Db=Da, and when Da=0, Db is also equal to zero, and the crossing is straight. The curve ordinates in either case will be equal to the ordinates between straight lines —that is to say, they will be one-fourth, one-half, and three-fourths of ($c-2g$), whether the rails AFa and BFb are straight or curved.

If, for instance, two seven-degree frogs be substituted for the six and eight degree frogs above, k would be the same as before, and equal to 37.2, and Db=Da=6° 00′.

The curve ordinates, obtained by subtracting from the ordinates in line 5, table IX., .37 times the corrections in table S, for a value of S equal to (Fb=7° 00′), would be 3.44, 2.30, and 1.15, equal to three-fourths, one-half, and one-fourth of the distance, ($c-2g$)=4.58.

These would also be the ordinates, at one-fourth, one-half, and three-fourths of (**AF**a), if AFa and BFa were straight lines, or when the degrees of curvature are zero, and, therefore, the rate of convergence between two curves with the same curvature is the same as when the radii are infinite and the curves coincide with their tangents.

PROBLEM XIX.

TANGENTIAL DEFLECTIONS BY SINGLE OR SIMPLE CURVES.

(*For demonstration see proposition XVI., article* 8.)

It is required to turn from one given tangential direction to another by means of a simple curve, and to determine the radius, or degree of curvature, and the positions of the tangent-points on the tangent-lines.

1. Measure the distance, (**AB**)=i, between any convenient points on the tangents, and also the acute angles, A′ and B′, between the line **AB** and the tangents through A and B, designating the greater of the two angles by the letter A′.

Determine the turn, or deflection, I, from one tangential direction to the other by the rules of problem V. The value of

this angle, according to the tangential directions given, will be as follows:

(Fig. 73.) $I=(A'+B')$. (Fig. 75.) $I=180°-(A'+B')$.
(Fig. 74.) $I=(A'-B')$. (Fig. 76.) $I=180°-(A'-B')$.

There are three methods of connecting the tangents: (I.) By a curve starting from one of the points A or B. (II.) By a curve starting at a given distance from one of those points. (III.) By a curve of given radius.

2. *Case* 1. *Required the radius of a curve through one of the tangent-points,* A *or* B, *and the position of its point of contact with the opposite tangent-line.*

Da is the degree, and a the radius, of the curve through A; Db is the degree, and b the radius, of the curve through B.

Tb is the point of contact of the curve through A with the tangent **BI**, and Ta the point of contact of the curve through B with the tangent **AI**. The distances from A to Ta, and from B to Tb, are equal, and are represented by the letter t.

3. *Solution by Construction* (figure 75).—From A and B as centres, with a radius, $\frac{1}{2}i \div \sin. \frac{1}{2}I$, describe two arcs intersecting at a point O. From O as a centre, with the same radius, describe a circle ATB cutting **AI** and **BI** at Ta and Tb. These will be the points of contact of the curves through A and B with the tangents **AI** and **BI**, and $(\mathbf{AT}a)=(\mathbf{BT}b)=t$.

From Tb, at right angles with **BI**, draw a line to intersection with the radius through A at a point C. $(\mathbf{CA})=a$ is the radius of the curve through A touching the tangent **BI** at Tb.

From Ta, at right angles to **AI**, draw a line to intersection with the radius through B at a point Z. $(\mathbf{ZB})=b$ is the radius of the curve through B touching the tangent **AI** at Ta.

4. *Formulæ.* The values of a, b, and t are determined as follows:

$$t=i.(\cos. B' \mp \cos. A') + \text{vers. } I = i.\sin. \tfrac{1}{2}(I \pm 2B) + \sin. \tfrac{1}{2} I. \quad (1)$$

$$\left.\begin{aligned} &a=i.\sin. B' + \text{vers. } I = t.\sin. B' + (\cos. B' \mp \cos. A') \\ &\text{Sin. } \tfrac{1}{2}Da = 50 \text{ vers. } I = i.\sin. B' = 50.(\cos. B' \mp \cos. A') + \\ &\qquad t.\sin. B'. \end{aligned}\right\} \quad (2a)$$

$$\left.\begin{aligned} &b=i.\sin. A' + \text{vers. } I = t.\sin. A' + (\cos. B' \mp \cos. A'). \\ &\text{Sin. } \tfrac{1}{2}Db = 50 \text{ vers. } I + i.\sin. A' = 50.(\cos. B' \mp \cos. A') + \\ &\qquad t.\sin. A'. \end{aligned}\right\} \quad (2b)$$

These equations take the following forms for different values of I:

Figure.	I	t	a	b
73	(A′+B′)	$\frac{i.\sin.\ ½\ (A'-B')}{\sin.\ ½\ (A'+B')}$	$\frac{i.\sin.\ B'}{1-\cos.\ (A'+B')}$	$\frac{i.\sin.\ A'}{1-\cos.\ (A'+B')}$
74	(A′−B′)	$\frac{i.\sin.\ ½\ (A'+B')}{\sin.\ ½\ (A'-B')}$	$\frac{i.\sin.\ B'}{1-\cos.\ (A'-B')}$	$\frac{i.\sin.\ A'}{1-\cos.\ (A'-B')}$
75	180°−(A′+B′)	$\frac{i.\cos.\ ½\ (A'-B')}{\cos.\ ½\ (A'+B')}$	$\frac{i.\sin.\ B'}{1+\cos.\ (A'+B')}$	$\frac{i.\sin.\ A'}{1+\cos.\ (A'+B')}$
76	180°−(A−B)	$\frac{i.\cos.\ ½\ (A'+B')}{\cos.\ ½\ (A'-B')}$	$\frac{i.\sin.\ B'}{1+\cos.\ (A'-B')}$	$\frac{i.\sin.\ A'}{1+\cos.\ (A'-B')}$

In these equations, the versed sine of I may be substituted for $1 \pm \cos.\ (A' \pm B')$.

Example 1 (figure 73). A′=20° 00′; B′=10° 00′; I=(A′+B′) =30° 00′; i=(**AB**)=1,000.

$$t = 1{,}000\ .\ \sin.\ ½\ 10° \div \sin.\ ½\ 30° = 336.7.$$
$$a = 1{,}000\ .\ \sin.\ 10° \div (1 - \cos.\ 30°) = 1296.1.$$
$$b = 1{,}000\ .\ \sin.\ 20° \div (1 - \cos.\ 30°) = 2552.9.$$

Example 2 (figure 74). Ta being in a pond, the tangent is produced, by triangulation, to A, which would be station 8.20 of the alignment.

Required the radius of a curve through B which will be tangent to the tangents **IA** and **IB**, and also the numbers of the stations at Ta and B.

The angles and distances are measured as follows: (**AB**)=i =300. A′= 66° 00′; B′=20° 00′. The angle IAB is obtuse, and therefore I=(A′−B′)=46° 00′.

$$t = 300\ .\ \sin.\ ½\ 86° \div \sin.\ ½\ 46 = 523.7.$$
$$b = 300\ .\ \sin.\ 66° \div (1 - \cos.\ 46°) = 897.6.$$

As Ta is 523.7 feet from A, which is at station 8.20, Ta is at station 2.963. Db = 6° 23′= 6.39° is the degree of curvature for the radius 897.6; the number of stations in the arc TaB is equal to I divided by the degree of curvature, or to 7.20, and, therefore, B is at station 2.963+7.20=10.163.

5. *Case* 2. *It is required to connect the two tangents by a curve starting at a given distance from* A *or* B.

In this case, r is the radius, and Ta and Tb are the tangent-points, of the connecting curve; I the intersection of the tangents; (**IA**) and (**IB**) the distances from I to the points A and B; (**IT**a) and (**IT**b) the equal distances from I to the points Ta and Tb, called the semi-tangents of the curve TaTb; (ATa) $= t_a$ the distance from A to Ta, and (**BT**b) $= t_b$ the distance from B to Tb.

Given the magnitude and direction of one of the distances, t_a *or* t_b*; required the magnitude and direction of the other distance, and the radius,* r*, of the connecting curve.*

Problem XII. (**IA**)$= i$.sin. B + sin. I. (1)

(**IB**)$= i$.sin. A + sin. I. (2)

Prop. XI. (1). (**IT**b)=(**IT**a)$=$
$$\left\{\begin{array}{l}(\mathbf{IA})+(\mathbf{AT}a)=i.\sin.\ \mathrm{B}' \\ \quad +\sin.\ \mathrm{I}+t_a. \\ (\mathbf{IB})+(\mathbf{AT}b)=i.\sin.\ \mathrm{A}' \\ \quad +\sin.\ \mathrm{I}+t_b.\end{array}\right\} \quad (3)$$

Prop. XI. (2).
$$\left\{\begin{array}{l}\text{When } t_a \text{ is given, } t_b=(\mathbf{BT}b)=(\mathbf{IT}b)- \\ \quad (\mathbf{IB})=i.\sin.\ \mathrm{B}'+\sin.\ \mathrm{I}+t_a-i.\sin.\ \mathrm{A}' \\ \quad +\sin.\ \mathrm{I}=t_a-\dfrac{i(\sin.\ \mathrm{A}'-\sin.\ \mathrm{B}')}{\sin.\ \mathrm{I}}. \\ \text{When } t_b \text{ is given, } t_a=(\mathbf{AT}a)=(\mathbf{IT}a)- \\ \quad (\mathbf{IA})=i.\sin.\ \mathrm{A}'+\sin.\ \mathrm{I}+t_b=i.\sin.\ \mathrm{B} \\ \quad +\sin.\ \mathrm{I}=t_b+\dfrac{i(\sin.\ \mathrm{A}'-\sin.\ \mathrm{B}')}{\sin.\ \mathrm{I}}.\end{array}\right\} \quad (4)$$

Prob. XIII. (10). $r = (\mathbf{IT}a)$. cot. $\frac{1}{2}\mathrm{I} = (\mathbf{IT}b)$. cot. $\frac{1}{2}\mathrm{I}$. (5)

As these sums and differences are algebraic, the signs of the terms are to be changed when they are negative, and therefore, (assuming distances measured in the direction IA or IB to be positive), in (3) change the sign prefixed to t_a when the directions (**IA**) and (**AT**a) are opposite, and the sign prefixed to t_b when the directions IB and ITb are opposite, and give to the semi-tangent, (**IT**a) or (**IT**b), in (4), the sign derived from (3).

In practice it may be determined from the figure whether the

semi-tangent is the sum or difference of (**IA**) and t_a, or (**IB**) and t_b, and whether t_b is the sum or difference of the semi-tangent and (**IB**), or t_a the sum or difference of the semi-tangent and (**IA**).

Example 1 (figure 73). A′=20°; B′=10°; I=(A′+B′)=30°.

It is required to determine the radius, r, of a curve connecting the tangents, and starting at a distance, (**AT***a*) = t_a = +200, from A, measured in the direction IA, when the distance, (**AB**)=i, is 1,000 feet.

(1) (**IA**)=i.sin. B÷sin. I = 1,000.sin. 10°÷sin. 30°= +347.3.

(3) Semi-tangent = (**IT***a*) = (**IT***b*) = (**IA**) + (t_a) = +547.3.

(2) (**IB**)=i.sin. A′÷sin. I=1,000.sin. 20°÷sin. 30°=+684.0.

(4) t_b=(**BT***b*) = (**IT***b*) − (**IB**)= +547.3−(+684.0)= −136.7.

(5) r = semi-tangent . cot. ½ I = 547.3 × cot. 15°=2042.5.

In (4), t_b is negative, and therefore it is to be measured in the direction BI, and T*b* will be between B and I.

Should it be required that the tangent-point T*a* should be 200 feet from A, measured in the direction AI, opposite to IA, the point T*a* would lie between I and A; the semi-tangent, derived from (3) by changing the sign of t_a, would be 147.3, and the radius would, in such case, be 147.3×cot. 15°=549.7.

6. *Case* 3. *Required the distances,* t_a *and* t_b, *from* A *and* B, *to the tangent-points,* T*a and* T*b, of a connecting curve of radius r.*

(5) (**IT***a*) = (**IT***b*) = r . tan. ½ I = semi-tangent. (6)

(1) and (2) (**IA**)= i. sin. B′÷sin. I. (**IB**)=i. sin. A′÷sin. I.

(4) t_a = (**AT***a*) = (**IT***a*) − (**IA**). t_b = (**BT***b*) = (**IT***b*) − (**IB**).

In (4), change the sign of (**IT***a*) when the directions IT*a* and IA are opposite, and the sign of (**IT***b*) when the directions IT*b* and IB are opposite. As the semi-tangent may be measured in either direction from I, there will be always two positions of the tangent-points T*a* and T*b*, for the same tangents and radius, corresponding to the pairs of curves shown in figures 73 to 76.

Example 1 (figure 74). A′=85°; B′=35°; i=170; r=680.4.

The interior angle, A, at the base of the triangle AIB, is obtuse, and therefore I=(A′− B′)=50°.

(6) Semi-tangent (**IT***a*)=(**IT***b*)=r. tan. ½I = 630.4 × tan. 25°
=307.3.
(**IA**) = i.sin. B′÷ sin. I = 127.3.
(**IB**) = i.sin. A′÷ sin. I=221.1.

When the semi-tangent is measured in the direction AI,

(4) t_a= −(**IT***a*)−(**IA**)= −307.3−127.3= −427.6.
t_b=(**IT***b*)−(**IB**)= +307.1−221.1= +86.2.

When the semi-tangent is measured in the direction IA,

t_a= +307.3−127.3= +200. t_b= −307.3−221.1= −528.4.

In the application of the formulæ, observe, when I is equal to 180°−(A′ ± B′), that tan. ½ I=cot. ½ (A′ ± B′), and cot. ½I= tan. ½ (A′ ± B′).

The only difficulty will be in the determination of the signs of the terms, and these may be more readily obtained by inspection of the figures, in any particular case, than by the rules.

PROBLEM XX.

(*For demonstration see proposition XVI., article VII.*)

From a point, A, *to produce or retrace a curve to a tangent-point,* T, *the tangent from which will pass through a given point,* B.

(Figures 71 to 76.) (**AB**) = i is the distance from A to B; A′ is the acute angle between **AB** and the tangent through A; T′ is the point on the curve through A from which the tangent produced will pass through B; B′ is the angle between this tangent and **AB**; t = (**TB**) is the length of the tangent, and Ia is the tangential deflection over the arc AT.

Case 1. *Given a,* (AB) = i, *and* A′; *required t,* B′, *and* Ia.

This and the preceding problem are special cases of problem XXVI. for connecting the tangents through A and B by means of two arcs with a common intermediate tangent, t.

In XIX., one of the radii is infinite, and the curve coincides with the tangent. In this problem, one of the radii becomes zero, or a point coinciding with the tangent-point, B, through which passes the extremity of the common tangent, *t*.

In both cases there will always be, for each value of the radius and common tangent, two positions of the tangent-point, T, and in the case under consideration these points may be determined by construction in the following manner:

(Figure 72.) C is the centre of the curve with the given radius, *a*, and B coincides with the centre of the curve for which the radius *b* is zero.

On (**CB**) as a diameter describe the circle CTB, and from C as a centre, with the given radius, *a*, describe an arc cutting this circle in two points, T and T'. The lines drawn from these points to B will be tangent to the curve through A, and equal to the tangent *t*. The equal deflections, right and left, from CB to the radii **CT** and **CT'**, are equal to the angle, T, obtained by the formula, tan. $T=t\div a$, and the sum of the deflections from A to the two tangent-points, T and T', is therefore equal to 2 T.

2. *Formulæ*. The values of *t*, B', and I*a* may be calculated as follows:

$$t=\sqrt{i.(i\pm 2\,a.\sin.\,A'.} \quad (1)$$

$$\text{Tan. } \tfrac{1}{2}\,B=\text{tan. } \tfrac{1}{2}\,A.(i-t)\div(i+t). \quad (2)$$

$$Ia=(A'\pm B'),\text{ or } 180°-(A'\pm B'). \quad (3)$$

In (1), for the double sign prefixed to 2 *a*.sin. A' substitute a plus sign when the curve is convex to **AB**, as in figure 72, and a minus sign when it is concave, as in figure 71. *t* will be greater than *i* when the curve is convex, and less when it is concave, and in the latter case the solution will not be possible if *i* is less than 2 *a*.sin. A.

In (2), A may be equal to the acute angle, A', or to the obtuse angle, 180°−A', and in the latter case tan. $\frac{1}{2}$ A = cot. $\frac{1}{2}$ A'.

There will always be two values of B corresponding to the two positions of the pair of equal tangents, and to the values, A' and 180°−A', of the angle A.

In (3), I*a* will be equal to the angle of intersection, I, determined by the rules of problem V., and equal either to (A'+B'), to (A'−B'), to 180°−(A'+B'), or to 180°−(A'−B').

In any of the formulæ above, B may be substituted for A, and b for a.

Example 1 (figure 71). (**AB**) $= i = 800$; $a = 573.7 =$ radius of a 10-degree curve; A′$=40°$; curve concave to **AB**.

In (1), the sign prefixed to 2 a.sin. A will be minus, because the curve is concave to **AB**, and therefore:

$2a = 1147.4$	log.	3.059704
Sin. (A′$=40°$)	"	9.808067
		2.867771 = log. (2 a.sin. A′$=737.6$).
$i - 2a$.sin. A$=800-737.6=62.4$	log.	1.795185
$i = 800$	"	2.903090
t^2	"	4.698275
$t=223.4$	"	2.349137

(2) Tan. $\frac{1}{2}$ B $=$ tan. $\frac{1}{2}$ A′.$(i-t)\div(i+t)$.

Tan. ($\frac{1}{2}$ A′ $= 20°$)	log.	9.561066
$i-t=576.6$	"	2.760875
$i+t=1023.4$	a.c. "	6.989950
		9.311891 = log. [tan. ($\frac{1}{2}$ B$=11°\ 35'$)]

(3) Ia=I=(A′+B′)$=40°\ 00'+23°\ 10'=63°\ 10'$.

For the second position of T, B and Ia will be as follows:

Tan. ($\frac{1}{2}$ B$=57°\ 03'$)=cot. ($\frac{1}{2}$ A′$=20°$).$(i-t)\div(i+t)$.
Ia=180°−(B−A)=180°−(114° 06′−40° 00′)=105° 54′.

Example 3 (figure 72¼). (**AB**) $= i = 400$; A′$=10°$; $a =$ radius of 6-degree curve $= 955.4$; curve convex to **AB**.

(1) $t = \sqrt{i.\ (i + 2a.\text{sin. A}')} = 541.1$.
(2) Tan. ($\frac{1}{2}$ B$=0°\ 45'$)=tan. ($\frac{1}{2}$ A′$=5°$).$(t-i)\div(t+i)=.0131$.
(3) Ia=(A′−B′)$=10°\ 00'-1°\ 30'=8°\ 30'$.

To obtain the tangent-point, T, the 6-degree curve must therefore be retraced $8°\ 30'$, or 1.42 stations.

Example 3 (figure 72). The values of B and Ia for the second tangent-point, T′, are obtained as follows:

(2) Tan. ($\frac{1}{2}$ B$=59°\ 44'$) = cot. ($\frac{1}{2}$ A′$=5°\ 00'$).$(t-i)\div(t+i)$.
(3) Ia=180°−(A′+B)$=50°\ 32$.

The angle B in this case is $119°\ 28'$, which is greater than shown in the figure, which is not drawn to scale.

The two tangent-points, T and T′, are obtained by running from A, right and left, 8° 30′ and 50° 32′, the length, t, being the same in each case. The sum, 59° 02′, of these angles is equal to twice the angle, T, calculated by the formula, tan. (T=29° 31′)=$t \div a$.

Case 2. *Given* t, (AB)$=i$, *and* A′; *required the radius* a, B′, *and* Ia.

$$a=\tfrac{1}{2}(i-t).\tfrac{1}{2}(i+t) \div \tfrac{1}{2}\, i.\text{sin. A.} \quad (4)$$

a being known, B and Ia are determined by (2) and (3) of case 1.

Example 1 (figure 71). Between T and B there is a bridge crossing to be placed on a tangent, and a length (t=223.4) is required for that purpose.

The line has been surveyed to a point A, 800 feet distant from B, and the tangent of the curve at A makes an angle of 40 degrees with the line **AB.**

Required the radius of a curve AT, of which the tangent, (TB)=t, shall pass through B and be equal to 223.4.

i=800; t=223.4; A′=40° 00′.

$\frac{1}{2}(i-t)$=288.3	log.	2.459845
$\frac{1}{2}(i+t)$=511.7	"	2.709015
$\frac{1}{2}\, i$=400 a.c.	"	7.397940
sin. (A′=40°) a.c.	"	0.191938
a=573.7	"	2.758738

This is the same with example 1 of case 1, except that t is given and a is required. B and Ia are determined by (2) and (3), case 1.

PROBLEM XXI.

COMPOUND CURVES WITH RADII OF MINIMUM DIFFERENCE.

(Figure 85.) Two tangents through A and B intersect at I; (**AB**)$=i$ is the distance from A to B; A′ and B′ are the greater and lesser acute angles between **AB** and the tangents through A and B; I$=$(A′$\pm$B′), or 180°$-$(A′$\pm$B′), is the turn, or deflection, from one tangential direction to the other, equal

to the algebraic sum, $Ia+Ib$, of the deflections over the two arcs; and a and b are their radii.

I, A′, B′, *and* i *are given; required the radii,* a *and* b, *of minimum difference, for compound curves.*

The formulæ are applicable to all values of I, but are not of any practical use except for cases when I is equal to (A′+B′) and the curves are as in figure 85, and in such case,

$$a=\frac{\frac{1}{2}\,i.\sin.\frac{1}{2}(A'-B')}{\sin.\frac{1}{2}I}.[\cot.\tfrac{1}{2}(A'-B')-\cot.\tfrac{1}{4}I]. \quad (1)$$

$$b=\frac{\frac{1}{2}\,i.\sin.\frac{1}{2}(A'-B')}{\sin.\frac{1}{2}I}.[\cot.\tfrac{1}{2}(A'-B')+\cot.\tfrac{1}{4}I]. \quad (2)$$

$$Ia=Ib=\tfrac{1}{2}I. \quad (3)$$

Example 1 (figure 85). $i=950$; $A'=8°\ 00'$; $B'=7°\ 00'$; $I=15°\ 00'$.

Cot. $[\frac{1}{2}(A'-B')=0°\ 30'']=114.599$
Cot. $(\frac{1}{4}I=3°\ 45')$ $=$ 15.257 diff., 99.342 ; sum, 129.856

$\frac{1}{2}i=475$. .....log.	2.676694		
Sin. $[\frac{1}{2}(A'-B')=0°\ 30']$. "	7.940842		
Sin. $(\frac{1}{2}I=7°\ 30')$ ...a.c. "	0.884202		
	1.501838		1.501838
99.342..... "	1.997124	129.856. .log.	2.113402
a=3155..... "	3.498962	b=4123.. "	3.615240

$Ia=Ib=\frac{1}{2}I=7°\ 30'$.

Example 2 (figure 85). $i=1{,}000$; $A'=20°$; $B'=10°$; $I=(A'+B')=30'$.

Cot. $[\frac{1}{2}(A'-B')=5°\ 00']=11.4301$
Cot. $(\frac{1}{4}I=7°\ 30')$.....= 7.5957 diff., 3.8344; sum, 19.0258

$\frac{1}{2}i=500$ .....log.	2.698970		
Sin. $[\frac{1}{2}(A'-B')=5°\ 00']$ "	8.940296		
Sin. $(\frac{1}{2}I=15°\ 00')$..a.c. "	0.587994		
	2.227260		2.227260
3.834..... "	0.583652	19.026...log.	1.279346
$a=647$..... "	2.810912	b=3201.. "	3.506606

$Ia=Ib=\frac{1}{2}I=15°\ 00'$.

It is best, when two curves are compounded, that the difference of the rates of curvature should be a minimum, unless the rate on one of the curves is thereby placed beyond the limit, or the nature of the ground prevents, in which cases one of the radii is assumed, and the other determined by problem XXII. As a rule, however, the limits of the radii within which the solution is possible for any particular form of curve are very narrow, and the formulæ of this problem are simple and easy of application, and should be used whenever circumstances permit.

Demonstration. By reference to the formulæ of table X., problem XXII., it will be seen, in the case of compound curves, that the difference of the radii is equal to a constant divided by the multiple, (sin. $\frac{1}{2}$ Ia) . (sin. $\frac{1}{2}$ Ib), and therefore the difference will be a minimum when the multiple is a maximum, which will be the case when the two factors are equal, or when Ia is equal to Ib. But as Ia+Ib=I, in such case $\frac{1}{2}$ Ia=$\frac{1}{2}$ Ib = $\frac{1}{4}$ I, and substituting these values of $\frac{1}{2}$ Ia and $\frac{1}{2}$ Ib in the formulæ for cot. $\frac{1}{2}$ Ia and cot. $\frac{1}{2}$ Ib, the values of a and b in (1) and (2) are deduced.

PROBLEM XXII. (TABLE X.)

COMPOUND AND REVERSED CURVES WITH COMMON TANGENT-POINTS.

1. A turn or deflection, I, is to be made, from one tangential direction to another, by means of two curves from tangent-points at A and B, compounding or reversing at a point of contact, T. A′ and B′ are the greater and lesser acute angles between **AB** and the tangent-lines through A and B, intersecting at I; Ia and Ib are the tangential deflections over the arcs AT and TB; a and b are the radii of the arcs, and (**AB**) = i is the distance from A to B.

I = (A′ ± B′), or 180° − (A′ ± B′)=Ia + Ib, is the total tangential deflection over the two arcs, determined by the rules of problem V.

The values of I for figures to which reference is made are given in table X. of this problem.

Given A′, B′, I, *i, and one of the radii, a or b; required* I*a*, I*b, and the other radius, b or a.*

2. *Solution by construction* (figure 78).—From A and B as centres, with a radius, $r = \frac{1}{2}i \div \sin. \frac{1}{2}$ I, describe two arcs intersecting at O. From O as a centre, with the same radius, describe a circle ATB cutting the tangents **AI** and **BI** at A, T*a*, B, and T*b*.

The circle ATB is the locus of the points of contact of all compound and reversed curves from the tangent-points A and B, for which the algebraic sum, I*a* + I*b*, of the tangential deflections is equal to I.

Representing the centres of the curves with radii *a* and *b* by C and Z, one of the radii, as *b*, may be determined, when the other radius, *a*, is given, in the following manner:

Produce the curve through A to intersection with the circle ATB at T. From T produce the radial line, **TC** or **CT**, to intersection with the radial line through B at a point Z. (**ZT**) = *b* will be the radius of the curve through B compounding or reversing with the curve through A.

When *b* is infinite, the curve coincides with the tangent **BI**, and cuts the circle ATB at a point T*b*, which is the point of contact of **BI** with the curve through A, the radius of which is determined by problem XIX. When the curve through A cuts the circle ATB at T_2, outside of the triangle AIB, the radial line $\mathbf{C_2T_2}$ cuts the radial line through B at a point Z on the opposite side of **AB** from O; the centres, C and Z, are also on the opposite sides of **AB**, and the curves are reversed, the difference, I*a*—I*b*, of the tangential deflections being, in this case, equal to I.

When the curve through A passes through B, $\frac{1}{2}$ I = A, $a = \frac{1}{2}\, i \div \sin. A = r$, the terminal tangent of the arc AB makes an angle $\frac{1}{2}$ (A′—B′) with the tangent **BI**, and *b* is equal to zero. For values of *a* greater than $\frac{1}{2}\, i \div \sin. A$, the curves will be compound, and the point of contact as shown in figure 86, the difference, I*a*—I*b*, of the tangential deflections being, in this case, equal to I.

When the radius *a* is infinite, the curve coincides with the tangent **AI**, and cuts the circle ATB at T*a*, which is the point of contact of the tangent **AI** with the curve through B, the radius *b* being determined by problem XIX. At T_3, where a

curve through B cuts the circle ATB, the centres, C_3 and Z_3, are on different sides of **AB**, and the curves are reversed. When the curve through B cuts the circle ATB at A, a is zero, and b is equal to $\frac{1}{2}i \div$ sin. B, the terminal tangent of the arc BA making an angle $\frac{1}{2}(A'-B')$ with the tangent **BI**.

It will be seen that the deflection, $Ia+Ib$, over each pair of these arcs is from the direction AI to IB, and equal to $(A'+B')$, the centres being on the same side of **AB** when the curves are compound, and on different sides when they are reversed, in accordance with the rules of problem V.

In figure 78, the deflection, $I=Ia+Ib$, over the arcs, is equal to $(A'+B')$, but the same rules will apply in the case of any other value of I. Thus, for instance, in figure 89 the curves are either compound with centres on different sides of **AB**, or reversed with centres on the same side, and therefore $I = 180° \pm (A'+B')$, and the radius of the circle ATB is equal to $\frac{1}{2}i \div$ sin. $\frac{1}{2}I = \frac{1}{2}i \div$ cos. $\frac{1}{2}(A'+B')$. In this figure, the curves touching at T_1 are reversed, with centres on the same side of **AB**; and as Ia and Ib are to the right, their sum, $Ia+Ib$, is equal to the right deflection, $180°-(A'+B')$, from AI to BI. The curves touching at T_2 are also reversed, with radii on the same side of **AB**, but the deflections, Ia and Ib, are both to the left, and their algebraic sum, $-Ia-Ib$, is equal to the left deflection, $-180°-(A'+B')$, from AI to BI.

The curves touching at T are compound, with centres on different sides of **AB**, and $-Ia+Ib = 180°-(A'+B')$.

In figures 82 and 83, $I=(A'-B')$; the radius of the circle ATB is $\frac{1}{2}i \div$ sin. $\frac{1}{2}(A'-B')$, and this circle is the locus of the points of contact of all curves from the tangent-points A and B, for which the algebraic sum, $Ia+Ib$, of the tangential deflections is equal to $(A'-B')$. In figure 83, $A'=B'$; $I=A'-B'=0$; $r=\frac{1}{2}i \div$ sin. ($\frac{1}{2}I=0$) is infinite; the circle ATB coincides with the chord **AB**, and the points of contact of all curves from the tangents at A and B will be on the line **AB**.

When the common tangent is a line, and not a point, the circle ATB will be the locus of the intersections of the chords produced. Thus, for instance, in figure 100 the chords **BT***b* and **T***a***A** intersect at K, on the circle ATB, the radius of which is equal to $\frac{1}{2}i \div$ sin. $\frac{1}{2}I$.

This class of curves is not, however, under consideration in this problem.

It may also be stated that the same principle is applied in problem XVI. to curves intersecting at an angle F, the radius of the circle AFB in that case being equal to $\frac{1}{2}i \div \sin. \frac{1}{2}(I + F)$.

3. *Formulæ.*—The formulæ for this problem are derived from those of table III., problem XVI., for intersecting curves, by making F equal to 0° or to 180°.

In this problem, as in problem XVI., the terms entering into the equations for different forms of curves differ only in their signs, and these signs, with the equations, are given in table X. for the curves represented in figures 78 to 88.

In these equations, the terms are so arranged as to give positive results for the class of curves to which they refer; and if the substitution of any given quantity in any equation produces a negative result, the solution is not possible for that value of the given quantity and the form of curves to which the equation is referred and the quantity must either be brought within the proper limits, or a form of curves adopted for which the equation will give positive results.

In the case of compound curves, as a rule, the radii for which the solution is possible in any particular case are contained within narrow limits, which are shown in table XII., problem XXVI., which applies whether the common tangent is a line or point; but the equations of this problem are simpler, and better adapted to compound curves when the common tangent is a point.

It will be seen by reference to the figures that there are two distinct classes of curves. In the one, both of the compound or reversed curves form a portion of the alignment; in the other, one of the curves is to be substituted for the other.

Thus, for instance, the two curves, AT and TB, would be on the alignment of a railroad-track in figures 83, 85, and 87, but in figures 80, 81, 81½, and 88 it is supposed that one of the curves, as that through A, has been located, and is to be retraced to a point, T, from which, as a tangent-point, another curve is to be substituted, terminating in a point, B, with a given direction of tangent.

In the latter case, one of the radii is always given, to determine the other; but in the former case both radii may be determined so as to fulfil any required conditions. The connection may be made, for instance, in the case of compound curves, by

curves with radii of minimum difference, as in problem XXI., or by means of a single curve, as in problem XIX., and, for reversed curves, by curves of equal radii, as in problem XXIII.

It will be seen that the deflection from one tangential direction to another may be always made by means of a variety of curves, but the particular form of curves to be adopted in any particular case will depend upon the following considerations: I. Whether the curves are to be compound or reversed. II. Whether each curve is convex or concave to **AB.** III. Whether the centres are on the same or different sides of **AB.**

The curve through one of the tangent-points, as A, is concave when it will cut the line **AB** if produced towards the opposite tangent-point, B, either between A and B or beyond B, as in figures 93 and 101. It is convex when the convexity is towards **AB,** as in figures 94 and 105.

In figures 83, 85, and 86, both curves are concave. In 86, the curve BT extends in the opposite direction from A, but is concave to **AB** if extended *towards* A. The curve through A is convex in figure 81, and concave in 81½, being the inside curve in one case and the outside in the other.

In every case the deflection, I, to be made over the curves is first to be determined by the rules of problem V.

It is to be noted, in all of the formulæ of table X., that A′ is the greater of the two acute angles, A′ and B′, and that a and Ia are the radius and deflection of the arc through A.

4. *Example* 1 (figure 85). $i = 950$; A′ = 8° 00′; B′ = 7° 00′; I = (A′ + B′) = 15° 00′; a = 3000; required b, Ia, and Ib.

The equations in table X. for figure 85 are as follows:

(1) Cot. ½ Ia = −[a.sin. ½ (A′+B′)]+[½ i.sin. ½ (A′−B′)] +cot. ½ (A′−B′).

a = 3000	log.	3.477121
sin. [½ (A′+B′)=7° 30′]	"	9.115698
½ i = 475 .... a.c.	"	7.323306
Sin. [½ (A′−B′)=0° 30′] a.c.	"	2.059158
94.468	"	1.975283

Cot. (½ Ia = 2° 51′) = + cot. [½ (A′ − B′) = 114.599] − 94.468 = + 20.131.

½ Ib = ½ I − ½ Ia = 4° 39′.

(2) $(b-a) = \frac{1}{2}i \,.\, \text{sin.}\, \frac{1}{2}(A'-B') \div (\text{sin.}\, \frac{1}{2}Ia \times \text{sin.}\, \frac{1}{2}Ib)$.

$\frac{1}{2}i \,.\, \text{sin.}\, \frac{1}{2}(A'-B') = 475 \,.\, \text{sin.}$

0° 30'..... log. 0.617536

Sin. ($\frac{1}{2}Ia$=2° 51')..... log. 8.695660

Sin. ($\frac{1}{2}Ib$=4° 39')..... " 8.909353 log. 7.605013

$b-a = 1029$. $b = a + 1029 = 4029$.

By problem XXI., the radii of minimum difference are, $a =$ 3155, and $b = 4123$. By problem XIX., $a = 3398$ is the radius of a curve tangent to **BI**, b being infinite. $a = 3413$ is the radius of the curve through A and B, and the corresponding value of b is 0, so that a change of 15 feet, in the value of a, changes b from infinity to zero.

For values of a between 3398 and 3413, the curves would be reversed.

Example 2 (figure 87). $A' = 30°\ 30'$; $B' = 4°\ 00'$; $I = (A'+B') = 34°\ 30'$; $a = 1000$; curves compound as in figure 85.

(1) Cot. ($\frac{1}{2}Ia = 23°\ 54'$) $= -a \,.\, \text{sin.}\, \frac{1}{2}(A'+B') \div \frac{1}{2}i \,.\, \text{sin.}\, \frac{1}{2}(A'-B') + \text{cot.}\, \frac{1}{2}(A'-B')$.

$\frac{1}{2}Ib = -\frac{1}{2}Ia + \frac{1}{2}I = -23°\ 54' + 17°\ 15' = -6°\ 39'$.

As this equation for figure 85 gives a negative value for $\frac{1}{2}Ib$, the solution is not possible for the assumed value of a, but the equation, ($\frac{1}{2}Ib = \frac{1}{2}Ia - \frac{1}{2}I$), for figure 87, gives a positive value; the curves will therefore be reversed, as in that figure, and the value of b is determined by the equation, $a+b = \frac{1}{2}i \,.\, \text{sin.}\, \frac{1}{2}(A'-B') \div \text{sin.}\, \frac{1}{2}Ia \,.\, \text{sin.}\, \frac{1}{2}Ib = 3175$, and $b = 3175 - a = 2175$.

In this case the maximum value of a, for compound curves, calculated by problem XIX., would be 516, but the tangents could be connected by a pair of reversed curves with a radius of 1059 feet and an intermediate tangent of 300 feet, as calculated in example 1, problem XXIII.

It will be seen that the only difference in the formulæ for figures 85 and 87 is in the sign of Ib, which changes the sign of b, the formula for $b - a$, in 85, being the same with the formula for $b + a$ in 87.

It will be seen that the curves in figures 80, 81, and 81½ are compound, with $I = (A' \pm B')$. In figures 80 and 81½, the curve AT is concave, and the curve BT is convex, but the

intersection, I, of the tangents through A and B, and the p. c. c., T, are on the same side of **AB** in figure 80, and on different sides in 81½. In 81, AT is convex and BT is concave, and I and T are on the same side of **AB.**

Example 3 (figure 80). $i=100'$; $A'=20°\ 00'$; $B'=10°\ 00'$; $I=(A'-B')=10°$; $a=2865$=radius of a 2-degree curve.

For the curve through A is to be substituted a curve TB, terminating at B with a tangential direction IB. Required the tangential deflections, I*a* and I*b*, and the radius, *b*, of the curve TB.

(1) $\text{Cot. } (\frac{1}{2}\,Ia = 3°\ 41') = +a \,.\, \sin.\ \frac{1}{2}\,(A'-B') + \frac{1}{2}\,i \,.\, \sin.\ \frac{1}{2}\,(A'+B') - \cot.\ \frac{1}{2}\,(A'+B').$
$\frac{1}{2}\,Ib = \frac{1}{2}\,Ia + \frac{1}{2}\,(A'-B') = 8°\ 41'.$

(2) $(a-b) = \frac{1}{2}i \,.\, \sin.\ \frac{1}{2}\,(A'+B') + \sin.\ \frac{1}{2}\,Ia \,.\, \sin.\ \frac{1}{2}\,Ib = 1335.$
$b = a - 1335 = 2865 - 1335 = 1530 =$ radius of a 3° 45′ curve.

The deflections, ($Ia=7.37°$), and ($Ib=17.37°$), divided by the degrees of curvature, ($Da=2.00°$) and ($Db=3.75°$), give the number of stations, 3.68 on AT, and 4.64 on TB.

Example 4 (figure 81). $i=300$; $A'=20°$; $B'=15°$; $I=(A'-B')=5°\ 00'$; $a=955.4$=radius of a 6-degree curve; required I*a*, I*b*, and *b*.

(1) $\text{Cot. } (\frac{1}{2}\,Ia = 13.43') = +a \,.\, \sin.\ \frac{1}{2}\,(A'-B') + \frac{1}{2}\,i \,.\, \sin.\ \frac{1}{2}\,(A'+B') + \cot.\ \frac{1}{2}\,(A'+B').$
$\frac{1}{2}\,Ib = \frac{1}{2}\,Ia - \frac{1}{2}\,(A'-B') = 11°\ 43'.$

(2) $(b-a) = \frac{1}{2}i \,.\, \sin.\ \frac{1}{2}\,(A'+B') + \sin.\ \frac{1}{2}\,Ia \,.\, \sin.\ \frac{1}{2}\,Ib = 976.7.$
$b = a + 976.7 = 1932.$ $Db = 8°\ 44'.$

Number of stations in AT=$Ia+Da=4.57$; number in TB = $Ib + Db = 7.63$.

Example 5 (figure 81½). Distances and angles as in last example.

(1) $\text{Cot. } (\frac{1}{2}\,Ia = 23°\ 59') = -a \,.\, \sin.\ \frac{1}{2}\,(A'-B') + \frac{1}{2}\,i \,.\, \sin.\ \frac{1}{2}\,(A'+B') + \cot.\ \frac{1}{2}\,(A'-B').$
$\frac{1}{2}\,Ib = \frac{1}{2}\,Ia - \frac{1}{2}\,(A'-B') = 21°\ 29'.$

(2) $(a-b) = \frac{1}{2}i \,.\, \sin.\ \frac{1}{2}\,(A'+B') + \sin.\ \frac{1}{2}\,Ia \,.\, \sin.\ \frac{1}{2}\,Ib = 308.1.$
$b = a - 308.1 = 656.3.$ $Db = 8°\ 44' = 8.73°.$

Number of stations in AT$=Ia+Da=7.59$. Number in TB $=4.88$.

Example 6. *Compound and Reversed Curves.* $I=180°-(A'+B')$ (figures 88 and 89).—The curves are compound, with centres on different sides of **AB**, in figure 88, and reversed, with centres on the same side, in figure 89; and therefore, by the rules of problem V., $I=Ia+Ib=180°-(A'+B')$ in both figures, and the formulæ are those of table X. for figures 88 and 89.

When, as is generally the case in practice, the exact relative positions of A and B are not of importance, and it is required to shift the tangent-point by a given distance, i=(**AB**), and to change the tangential direction by a given angle, I, the following approximate formulæ will be found sufficiently exact, when i does not exceed 50 or 60 feet, and I is not greater than 15 or 20 degrees.

The change, I, of tangential direction is determined according to the rules of problem III., as the algebraic difference of the deflections to the two tangents from any third direction AB, or in most cases, with sufficient accuracy, as the algebraic difference of their magnetic bearings.

The number, k, of stations in the arc AT, which is also equal to the number in the arc TB, and the change of the degree of curvature, may be determined by the following formulæ, deduced from (40), article 5, proposition XV., by making the frog-angle, $F'=0°$, under the condition that the lengths of the arcs AT and TB, or the number of stations in the arcs, shall be equal.

Da and Db are the degrees of the two arcs, and when the curves are compound, as in figure 88, Db is always the degree of the *inside curve*, and greater than Da. T is the point of compound or reversed curvature, and, as before stated, k is the number of stations in each of the arcs, AT and TB.

(Db ± Da) is the change of the degree of the located curve, equal to (Db+Da) for reversed curves, and to (Db—Da) for compound curves.

The values of (Db ± Da), and of k, are independent of the values of Da and Db, and constant for the same values of i and I.

$$(Db \pm Da)=.874\times I^2+i=I+k. \quad \text{(a)}$$

$$k=1.144\times i\div \mathrm{I}=\mathrm{I}\div(\mathrm{D}b\pm \mathrm{D}a). \quad \text{(b)}$$

$$\text{For compound curves, } \begin{pmatrix} \mathrm{D}b=.874\times \mathrm{I}^2\div i+\mathrm{D}a. \\ \mathrm{D}a=\mathrm{D}b-.874\times \mathrm{I}^2\div i. \end{pmatrix} \quad \text{(c)}$$

$$\text{For reversed curves, } \begin{pmatrix} \mathrm{D}b=.874\times \mathrm{I}^2\div i-\mathrm{D}a. \\ \mathrm{D}a=.874\times \mathrm{I}^2\div i-\mathrm{D}b. \end{pmatrix} \quad \text{(d)}$$

In (c), when $\mathrm{D}b$ is less than $.874\times \mathrm{I}^2\div i$, $\mathrm{D}a$ is negative and the curves reversed.

In (d), if $\mathrm{D}a$ or $\mathrm{D}b$ is greater than $.874\times \mathrm{I}^2\div i$, the result is negative and the curves compound.

Example. The tangent-point is to be shifted a distance, $i=$ (**AB**)$=20$ feet; the tangential direction, at the terminus of the located curve, bears N. 3° W., and it is required to change to a tangential direction bearing N. 5° 48′ E.

Problem III. $\mathrm{I}=+5°\ 48'-(-3°\ 00')=8°\ 48'=8.80°$.

(a) $(\mathrm{D}b\pm \mathrm{D}a)=.874\times \mathrm{I}^2\div i=.874\times (8.8)^2\div 20=3.38°=3°\ 23'$.

(b) $k=1.144\times i\div \mathrm{I}=1.144\times 20\div 8.80=2.60$.

k may be calculated first, and then $(\mathrm{D}b\pm \mathrm{D}a)=\mathrm{I}\div k=8.80°\div 2.60=3.38$.

Whatever may be the degree of the located curve, it must, therefore, be retraced 2.60 stations to the point, T, from which 2.60 stations of another curve, differing in degree by 3° 23′, must be substituted, and, in such case, the terminal points, A and B, of the two curves will be 20 feet apart, with a difference of 8° 48′ in the tangential directions.

$$\left.\begin{array}{l} \text{For compound curves, if } \mathrm{D}a=4°\ 00', \mathrm{D}b=3°\ 23'+4°\ 00'=7°\ 23'. \\ \text{For reversed curves, if } \mathrm{D}a=4°\ 00', \mathrm{D}b=3°\ 23'-4°\ 00=-0°\ 37'. \end{array}\right\} \quad (1)$$

In such case the negative value of $\mathrm{D}a$ shows that the curves cannot be reversed for a value of $\mathrm{D}a=4°\ 00'$.

$$\left.\begin{array}{l} \text{For compound curves, if } \mathrm{D}b=4°\ 00', \mathrm{D}a=4°\ 00'-3°\ 23'=+0°\ 37'. \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l} \text{For compound curves, if } \mathrm{D}b=3°\ 23', \mathrm{D}a=3°\ 23'-3°\ 23'=0°\ 00'. \end{array}\right\} \quad (3)$$

In this case the curve through A is straight and coincides with its tangent.

For reversed curves, if $Db = 2°\ 23'$, $Da = 3°\ 23' - 2°\ 23' = 1°\ 00'$. (4)

In the above examples, the tangential deflections, Ia and Ib, over the arcs AT and TB, each of 2.60 stations, are as follows:

(1) $Ia = Da \times 2.60 = 4°\ 00' \times 2.60 = 10°\ 24'$.
$Ib = Db \times 2.60 = 7°\ 23' \times 2.60 = 19°\ 11'$. $I = Ib - Ia = 8°\ 47'$.

(2) $Ia = Da \times 2.60 = 0°\ 37' \times 2.60 = 1°\ 36'$.
$Ib = Db \times 2.60 = 4°\ 00' \times 2.60 = 10°\ 24'$. $I = Ib - Ia = 8°\ 48'$.

(3) $Ia = Da \times 2.60 = 0°\ 00' \times 2.60 = 0°\ 00'$.
$Ib = Db \times 2.60 = 3°\ 23' \times 2.60 = 8°\ 47'$. $I = Ib - Ia = 8°\ 47'$.

(4) $Ia = Da \times 2.60 = 1°\ 00' \times 2.60 = 2°\ 36'$.
$Ib = Db \times 2.60 = 2°\ 23' \times 2.60 = 6°\ 11'$. $I = Ib + Ia = 8°\ 47'$.

It will be seen that the algebraic sum, $Ia + Ib$, of the tangential deflections, is the arithmetic difference when the curves are compound, and the arithmetic sum when they are reversed.

In the preceding examples, the points A and B are equidistant from T, and in the triangle ATB the chord $(\mathbf{TA}) = (\mathbf{TB}) = k \times 100 = 260$; the angle ATB between the chords $= \frac{1}{2} I = 4°\ 24'$, and the exact value of the base, $(\mathbf{AB}) = i$, is determined as follows:

$$i = 2\ k \times 100 \times \sin.\ (\tfrac{1}{2}\ \mathrm{ATB} = \tfrac{1}{4}\ \mathrm{I}) = 320 \times \sin.\ 2°\ 12' = 19.97.$$

To insert an intermediate tangent, t, between the curves when they are reversed, see formulæ of table XII., problem XXVI., for figure 107.

Example 7. *Curves, compound or reversed, with parallel tangents at the extremities,* A *and* B, (figures 81¼ and 83).—If T is the point of compound or reversed curvature, then, by problem VII., figures 44 and 45, the lines of the chords coincide, and A, T, and B are on the same straight line; the tangential deflections, Ia and Ib, over the arcs TA and TB, are equal to each other and to half the tangential deflection, Ia or Ib, and the deflection $I = Ia + Ib = 0°$.

Case 1. If it be required that the curves shall terminate at two fixed points, A and B, measure the distance, $i = \mathbf{AB}$,

and the angle A′=B′ between the line **AB** and the parallel tangents.

Given A′=B′, *and one of the radii, a or b; required the other radius, and the equal tangential deflections,* I*a and* I*b, from* A *and* B *to the point,* T, *of compound or reversed curvature.*

$$\tfrac{1}{2}\,\mathrm{I}a=\tfrac{1}{2}\,\mathrm{I}b=\mathrm{A}'=\mathrm{B}'. \quad \text{(a)}$$

$$(a\pm b)=\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a=(\text{nearly})\ i\times 57.2\div\mathrm{I}a. \quad \text{(b)}$$

In (b), substitute $(a+b)$ for $(a\pm b)$ when the curves are reversed, and $(a-b)$ when they are compound, and, in the latter case, *the longer* radius, a, is always to be the radius of the outside curve.

Equation (b) takes the following forms for reversed and compound curves:

$$\text{For compound curves,} \begin{cases} a=\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a+b. \\ b=a-\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a. \end{cases} \quad \text{(c)}$$

$$\text{For reversed curves,} \begin{cases} a=\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a-b. \\ b=\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a-a. \end{cases} \quad \text{(d)}$$

Any value of a or b which, substituted in (c) or (d), gives a negative value to b or a, will change the curves from compound to reversed, or from reversed to compound.

Equation (b) is deduced from the formulæ of table X. for $(a\pm b)$, in the cases of figures 81¼ and 83, by substituting the values of I*a* and I*b* from (a).

Example. The distance i=(**AB**)=40′, and the angle **A′=B′**, between (**AB**) and the parallel tangents, is 6° 00′.

(a) $\tfrac{1}{2}\,\mathrm{I}a=\tfrac{1}{2}\,\mathrm{I}b=\mathrm{A}'=6°\ 00'$.

(b) $(a\pm b)=\begin{cases} \tfrac{1}{2}\,i\div\sin.\ \mathrm{A}'=\tfrac{1}{2}\,i\div\sin.\tfrac{1}{2}\,\mathrm{I}a=20\div\sin.\ 6°=191.8. \\ i\times 57.2\div\mathrm{I}a=40\times 57.2\div 12=190.7\ \text{(nearly).} \end{cases}$

This is the change in the radial value, and is independent of the value of the given radius, and constant for the same values of i and A′.

For compound curves, if b=955.4, (c), $a=\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a+b=$ 955.4+191.3=1146.7.

For compound curves, if a=955.4, (e), $b=a-\tfrac{1}{2}\,i\div\sin.\ \tfrac{1}{2}\,\mathrm{I}a=$ 955.4−191.3=764.1.

For reversed curves, if $b=955.4$, (d), $a=\frac{1}{2}i\div\sin.\ \frac{1}{2}\mathrm{I}a-b=191.3-955.4=-764.1$.

Reversed curves are not, therefore, possible for a value $b=955.4$, or for any value of a or b greater than 191.3.

If the curves are reversed, and $a=b=r$,

$$r=\tfrac{1}{4}i\div\sin.\ \tfrac{1}{2}\mathrm{I}a=\tfrac{1}{4}i\div\sin.\ \mathrm{A}'. \quad \text{(e)}$$

Of the three quantities $(a\pm b)$, i, and $\mathrm{I}a$, two being given, the third is determined by (b).

Case 2. As a rule, *for compound curves at least*, it is required that the perpendicular distance between the parallel tangents shall be equal to a given distance, represented, in the formulæ below, by p; and one of the radii, a or b, and the tangential deflection, $\mathrm{I}a$ or $\mathrm{I}b$, over the arc AT or BT, is given, to find the other radius.

$$(a\pm b)=p\div\text{vers. }(\mathrm{I}a=\mathrm{I}b)=, \text{ nearly, } 6600\times p\div\mathrm{I}a^2. \quad \text{(f)}$$

For compound curves, $\begin{cases} a=p\div\text{vers. } \mathrm{I}a+b. \\ b=a-p\div\text{vers. } \mathrm{I}a. \end{cases}$ (g)

For reversed curves, $\begin{cases} a=p\div\text{vers. } \mathrm{I}a-b. \\ b=p\div\text{vers. } \mathrm{I}a-a. \end{cases}$ (h)

In (g), a is the radius of the *outside* curve, and a negative result in (g) or (h) will show that the solution is not possible for the given radius and form of curves.

Example. Compound curves.—A 5° curve, radius $a=1146.7$, has been run to a point A, with a tangential direction AI; required the radius, b, of an inside curve, which, starting at a point T, 2.40 stations, or 12 degrees, back on the 5° curve, with a tangential deflection, $\mathrm{I}b=12°\ 00'$, shall terminate in a tangent parallel to AI, and at a perpendicular distance, $p=4.18$, from it.

(g) $b=\begin{cases} a-p\div\text{vers. } \mathrm{I}a=1146.7-191.3=955.4. \\ a-6600\times 4.18\div 12^2=1146.7-191.5=955.2, \text{ nearly.} \end{cases}$

If a and b are given $\left\{\begin{array}{l} \text{vers. } \mathrm{I}a=p\div(a\pm b). \\ \mathrm{I}a=\sqrt{6600\times p\div(a\pm b)}, \text{ nearly.} \end{array}\right\}$ (i)

The formulæ for reversed curves will apply to crossings between parallel tracks, when p is the perpendicular distance between the lines of *parallel frog-tangents.*

PROBLEM XXIII. (TABLE XI.)

REVERSED CURVES WITH EQUAL RADII.

To connect two tangent-points, A *and* B, *by a pair of reversed curves with an intermediate tangent.*

(**AB**)$=i$ is the distance from A to B ; A′ and B′ are the greater and lesser acute angles between **AB** and the tangent-lines **AI** and **BI.**

I$=$(A′$\pm$B′), or 180° $-$ (A′$\pm$B′), determined by problem V., is the algebraic sum, Ia+Ib, of the tangential deflections over the two curves, from the initial to the terminal tangential direction.

r is the common radius, and t the length of the intermediate tangent.

In table XI. are given the formulæ for different values of I, table A applying whether t is a line or point, and table B only when it is a point of contact. The formulæ of table B are simpler than those of table A, but, as a rule, the latter should be used.

The formulæ of table XI. are deduced from those of table XII. by making a equal to b, and the formulæ of table B in the same manner from table X.

2. *Example* 1 (figures 102 and 87). A′=30° 30′; B′=4° 00′; (**AB**)$=i$=1300. The triangle AIB is similar to figure 7; the centres of the curves are on different sides of **AB,** and the curves are reversed, and therefore I is equal to (A′+B′), and the formulæ are those of table A, figure 102, or table B, figure 87.

Suppose, first, that the intermediate tangent is t=300.

(I.) Determine R by formula (1), table XI., as follows :

(1$-t+i$)=(1$-$300+1300)=.7692.....	log.	9.886089
(1+$t+i$)=(1+300+1300)=1.2308.....	"	9.090188
1$-t^2+i^2$=R)=.9467.....	"	9.976277

(II.) Determine the value of S and S^2 by formula (2), as follows:

½ (sin. A′−sin. B′)=.21889	log.	9.340226
Sin. [½ (A′+B′)=17° 15′]	"	9.472086
S=.7383	"	9.868140
S^2=.5449	"	9.736280
R+S^2=1.4916	"	0.173671
$\sqrt{R+S^2}$=1.4916=1.2213	"	0.086835

(III.) Determine the value of r by formula (3), as follows:

$\sqrt{R+S^2}$−S=.4830	log.	9.683948
½ i=650	"	2.812913
Sin. (½ I=17° 15′) a.c.	"	0.527914
r=1058.6	"	3.024775
(4) ½ t=150	"	2.176091
Tan. (T=8° 04′)	"	9.151316
(5) ½ (cos. B′+cos. A′)=.92959	"	9.968296
Cos. (T=8° 04′)	"	9.995680
Cos. (C=23° 01′)	"	9.963976

(6) Ia=(C−T)+A′=45° 27′. Ib=(C−T)−B=10° 57′.
Algebraic sum, Ia−Ib=I=34° 30′.

If t is zero, the calculation may be made by the same formulæ, substituting 0 for t, but with less calculation by table B as follows:

(1) cos. (C=21° 38′)=½ (cos. B′+cos. A′)=.92959.
(2) Ia=C+A′=52° 08′. Ib=C−B′=17° 38′.

(3) ¼ i=325			log.	2.511883
Sin. [½ (A′−B′)=13° 15′]			"	9.360215
Sin. (½ Ia=26° 04′)	log.	9.642877	"	1.872098
Sin. (½ Ib=8° 49′)	"	9.185466	"	8.828343
r=1106			"	3.043745

The squares and square roots, calculated above by logarithms, may be taken from a table of squares.

The reduction of the radius by inserting an intermediate tangent of 300 feet is 47 feet. By problem XIX. the maximum value of the radius, for a connection by means of a compound

curve, would be 516 feet, corresponding to a degree of curvature of 11° 06′.

Example 2 (figures 101 and 83). $i=600$; $A'=18°$; $B'=6°$.

In this case the triangle AIB is similar to figure 7a; the curves are reversed, and the centres on different sides of **AB**, and, therefore, $I=(A'-B')=12°$, and the solution will be by the formulæ in table XI. for figures 101 and 83.

If a tangent, $t=300$, is to be inserted between the two branches, calculate by formulæ (1) to (6), table A.

(1) $R=(1-t^2\div i^2)=.75$. (2) $S=\frac{1}{2}$ (sin. A + sin. B) + sin. $\frac{1}{2}$ I=1.9782. (3) $r=\frac{\frac{1}{2}\,i}{\text{sin. }\frac{1}{2}\text{ I}}\times(\sqrt{R+S^2}-S)=520$. (4) Tan. ($T=16°\ 05'$) $=\frac{1}{2}\,t\div r$. (5) cos. ($C=20°\ 49'$)=cos. T . (cos. B′+cos. A′). (6) $Ia=(C-T)+A=22°\ 44'$; $Ib=C-T+B=10°\ 44'$. $Ia-Ib=I=12°\ 00'$.

If $t=0$, calculate by (1), (2), and (3), table B, as follows: (1) cos. ($C=13°\ 24'$)=$\frac{1}{2}$ (cos. B′+cos. A′). (2) $Ia=C+A=31°\ 24'$. $Ib=C+B=19°\ 24'$. (3) $r=\frac{1}{4}\,i$. sin. $\frac{1}{2}$ (A′+B′)+ sin. $\frac{1}{2}$ Ia . sin. $\frac{1}{2}$ Ib=684.0. In this case the degree of curvature is increased from 8° 34′ to 11° 02′, to insert the 300-feet tangent.

Example 3. *Reversed curves between parallel tangents* (figures 83 and 101).—When $A'=B'$, and $I=(A'-B')=0°$, the formulæ of table X. are indeterminate, as the terms which enter into the equations have a factor equal to zero in the numerator and denominator, and the solution will be by the following formulæ:

(Figure 101) $r=\frac{1}{2}\,(i-t)\,.\,\frac{1}{2}\,(i+t)\div i$. sin. A. Tan. $T=\frac{1}{2}\,t\div r$. Cos. C=cos. T.cos. A′. $Ia=Ib=C-T+A'$.

(Figure 83) If $t=0$, $Ia=Ib=2$ A. $r=\frac{1}{4}\,i\div$sin. A.

If $A'=B'=12°$, $I=(A'-B')=0°$, $t=300$, and $i=1300$. $r=1480$; $T=5°\ 47'$; $C=13°\ 18'$; $Ia=Ib=19°\ 31'$.

If t is equal to zero, $r=1563$; $Ia=Ib=24°$.

In figures 105 and 107, one of the radii is given, and if the other is to be equal, t must be determined by problem **XXV.**

PROBLEM XXIV.

(*See proposition XVI., article* 4, *for demonstration.*)

1 (Figures 93 and 94). A and B are tangent-points on two circles; (**AB**)=i is the distance from A to B; A′ and B′ are the greater and lesser acute angles between **AB** and the tangents **AI** and **BI**; a and b are the radii through A and B; C and Z are the centres of the circles; c=(**CZ**) is the distance from C to Z.

Either curve, as that through A, is concave, or convex, when the extension towards the opposite tangent-point, B, is concave, or convex, to the line **AB.** In figure 93, both curves are concave; in figure 94, the curve through A is convex and the curve through B concave; in figure 101, both curves are concave; in figure 105, both are convex; in figure 86, both curves are concave, for although the curve BT is convex when extended in the direction AB, it is concave when extended in the opposite direction, BA.

The centres are on different sides of **AB** in 94, and on the same side in 93. For the purposes of this problem it is to be specially noted whether the curves are concave or convex, and whether their centres are on the same or different sides of **AB.**

Given these characteristics, and also i, a, b, A′, *and* B′; *required the distance,* c, *from* C *to* Z, *and the deflections,* Ca *and* Cb, *over the curves, from* A *and* B, *to the points at which* **CZ** *cuts the arcs of the two circles.*

2. Assume an axis of direction **CN,** from the centre C, towards and at right angles to the line **AB,** and consider the direction **AB,** whether to the right or left, looking from CN, as positive.

The origin of co-ordinates, C, may be the centre of either circle. In this problem it is the centre of the curve through A. As a rule it is convenient to assume C to be the centre of the curve of longer radius. The line (**CZ**) is the closing line to the three lines (**CA**) = a, (**AB**)=i, and (**BZ**)=b, the first estimated from centre to tangent, and the last from tangent to centre. C=CN . CZ is the bearing of the closing line; CN.CA =A, of the radius a; CN.AB=+90°, of **AB,** and CN.BZ of

the radius **BZ.** The bearing and length of the closing line are to be calculated by departures and latitudes, according to the rules of problem XI. and proposition XIII.

3. *Bearings, from* CN, *of* CA, AB, *and* BZ.

The bearings, A and B, of CA and BZ, are either the angles A′ and B′, or their supplements, and their signs and magnitudes are determined by the following rules:

$$\left.\begin{array}{l} \text{A} = \begin{cases} +\text{A}' \text{ when the curve through A is convex to } \mathbf{AB}. \\ -\text{A}' \quad \text{“} \quad\quad \text{“} \quad\quad \text{“ concave “} \end{cases} \\ \text{B} = \begin{cases} \text{B}' \text{ when the centres are on the same side of } \mathbf{AB}. \\ 180° - \text{B}' \quad \text{“} \quad \text{“} \quad \text{“ different sides “} \end{cases} \\ \text{B is} \begin{cases} \text{positive when the curve through B is convex to } \mathbf{AB}. \\ \text{negative} \quad \text{“} \quad\quad \text{“} \quad\quad \text{“} \quad \text{“ concave “} \end{cases} \end{array}\right\} \quad (1)$$

CN.AB is, by supposition, positive and equal to $+90°$.

4. If Σd is the departure, and Σl the latitude, of the closing line, **(CZ),**

$$\Sigma d = +i \pm a.\sin.\text{A}' \pm b.\sin.\text{B}'. \quad \Sigma l = +a.\cos.\text{A}' \pm b.\cos.\text{B}'. \quad (2)$$

In these equations, i is always positive; $a.\sin.\text{A}'$, or $b.\sin.\text{B}'$, is positive when the curve through A or B is convex, and negative when it is concave ; $a.\cos.\text{A}'$ is always positive, and $b.\cos.\text{B}'$ is positive when the centres are on different sides of **AB,** and negative when they are on the same side.

5. *Bearing and length of the closing line* **(CZ)**$=c$.

Σd and Σl are the sides of a right-angled triangle of which **(CZ)** is the hypothenuse ; and the length, c, and the bearing, C, are determined either by (3) and (4), or by (5) and (6), as follows:

$$\text{Tan. C} = \Sigma d \div \Sigma l. \quad (3)$$
$$c = \Sigma d \div \sin.\text{C} = \Sigma l \div \cos.\text{C}. \quad (4)$$
$$c^2 = (\Sigma d)^2 + (\Sigma l)^2. \quad (5)$$
$$\text{Sin. C} = \Sigma d \div c. \quad \cos.\text{C} = \Sigma l \div c. \quad (6)$$

In these equations, if C′ is the acute angle corresponding to the numerical value of the tangent, sine, or cosine, C is equal to C′ when Σl is positive, and to $180° - \text{C}'$ when Σl is negative, and has always the same sign with the departure, Σd.

6. *To determine the deflections,* C*a and* C*b.*—C*a* and C*b* are

the points at which the line **CZ** cuts the curves through A and B, and C*a* and C*b* are the tangential deflections over the arcs AC*a* and C*b*B. By proposition X., the tangential deflection over an arc is equal to the deflection between the radii through its extremities, provided that the radial directions are both estimated from centre to tangent, or from tangent to centre. C*a* is, therefore, always equal to CA.CZ, and C*b* to CZ.BZ except when the centres are on the same side of **AB**, in which case C*b* is equal to 180°—CZ.BZ.

By proposition V., the deflection from any direction to another is equal to the algebraic difference of their bearings from any third direction, and, therefore,

$$C\alpha = CA.CZ = CN.CZ - CN.CA = C - A. \quad (7)$$

$$C\beta = CZ.BZ = CN.BZ - CN.CZ = B - C \text{ for centres on different sides of } \mathbf{AB}, = 180° - (B - C) \text{ for centres on the same side of } \mathbf{AB}. \quad (8)$$

In these equations, A, B, and C must have the signs and values determined by (1) and (3), and the sums and differences are algebraic.

If the calculated value of C*a* or C*b* is greater than 180 degrees, reduce to an equivalent deflection less than 180 degrees by the rules of problem II. In making this reduction, the sign prefixed to 180 degrees, when it is a term of the equation, may be changed from plus to minus or from minus to plus.

The line **CZ** cuts each of the curves through A and B at two diametrically opposite points, of which one is determined by the equations above, and the other is obtained by adding or subtracting 180 degrees.

7. *Example* 1 (figure 83). $i = 600$; $A' = 18°\ 00'$; $B' = 6°\ 00'$; $a = b = 684$; both curves concave, and with centres on different sides of **AB**.

(2) $\Sigma d = +i - a.\sin. A' - b.\sin. B' = 600 - 211.4 - 71.5 = +317.1$

$\Sigma l = +a.\cos. A' + b.\cos. B' = 650.5 + 680.3 = +1330.8$

In these equations, the departures, $a.\sin. A'$ and $b.\sin. B'$, are negative because the curves are each concave, and the latitude, $b.\cos. B$, is positive because the centres are on different sides of **AB**.

(3) Tan. $(C=+13^\circ\ 24')=\Sigma d \div \Sigma l=+317.1\div(+1330.8)=.2382$.

In this equation, C is acute because Σl is positive, and has the same sign with Σd.

(4) $c=\Sigma d \div \sin. C=\Sigma l \div \cos. C=1368$.

(1) $A=-A'=-18^\circ\ 00'$, because the curve through A is concave.

$B=-B'=-6^\circ\ 00'$, because the curve through B is concave and the centres are on different sides of **AB**.

(7) $Ca=C-A'=(+13^\circ\ 24')-(-18^\circ\ 00')=+31^\circ\ 24'$.

(8) $Cb=B-C=(-6^\circ\ 00')-(+13^\circ\ 24')=-19^\circ\ 24'$.

In this example, the radii are equal, and the calculated value of c is equal to their sum, and therefore the curves touch and are reversed, as in example 2, problem XXIII., in which the values of A', B', and C are the same as in this example, and the curves are reversed with equal radii.

Example 2 (figure 93). The signs and values of the quantities are the same as in the preceding example, except that the centres are on the same side of **AB**.

(2) $\Sigma d=+i-a.\sin. A'-b.\sin. B'=600-211.4-71.5=+317.1$.

$\Sigma l=a.\cos. A'-b.\cos. B'=650.5-680.3=-29.8$.

(3) Tan. $(C=+95^\circ\ 22')=+317.1\div(-29.8)=10.641$.

In this case, the acute angle, C', corresponding to the tangent 10.461, is $84^\circ\ 38'$, and C is equal to $180^\circ-C'$, because Σl is negative, the sign being the same with Σd.

(4) $c=\Sigma d \div \sin. C=\Sigma l \div \cos. C=318.5$.

(1) $A=-A'=-18^\circ\ 00'$, because the curve through A is concave.

$B=-(180^\circ-B')=-174^\circ\ 00'$, because the curve through B is concave and the centres are on the same side of **AB**.

(7) $Ca=C-A=+95^\circ\ 22'-(-18^\circ\ 00')=+113^\circ\ 22'$.

(8) $Cb=180^\circ+(B-C)=180^\circ+(-174^\circ\ 00'-95^\circ\ 22')=-89^\circ\ 22'$.

These are the values of Ca and Cb corresponding to one of the points at which **CZ** cuts the circles. But **CZ** cuts the circles at two other points diametrically opposite to Ca and Cb, and the corresponding values of Ca and Cb are, $-180+(+113^\circ\ 22')=-66^\circ\ 38'$, and $+180^\circ+(-89^\circ\ 22')=+180^\circ-89^\circ\ 22'=+90^\circ\ 38'$.

In both of these examples the origin of co-ordinates, C, is assumed at the centre of the curve with radius a, and as B is right of A, the right is positive. Assuming CN as north, Σd will be positive when Z is to the right of **CN**, and negative when it is to the left, as in figure 85, or rather as it would be in that figure if **CN** were drawn from the centre of the curve with radius a.

For values of Σd, Σl, and C, for figures to which reference is made, see table XII., problem XXV.

PROBLEM XXV. (TABLE XII.)

(For demonstration see proposition XVI., article 5.)

The radii and positions of two circles are given, and it is required to determine the lengths and positions of the common tangents connecting the curves.

1. *Solution by construction* (figures 97 and 99).—C and Z are the centres of the circles; a and b their radii; $(a \pm b)$ the sum of the radii when the curves are reversed, and the difference when they are compound, and $c=$(**CZ**) the distance between the centres.

On (**CZ**) as a diameter describe a circle CPZ.

From one of the centres, as C, with a radius $(a \pm b)$, describe an arc cutting the circle CPZ in two points, P and P'.

From C, on the lines CP and CP', draw the radii (**CT**a) and (**CT**'a) to the curve through A. From Z, parallel to **CT**'a and **CT**a, draw the radii (**ZT**b) and (**ZT**'b) to the curve through B. (**T**a**T**b) and (**T**'a**T**'b) are the required tangents connecting the curves, and they are of equal length. If t is the length of the pair of common tangents,

$$c^2=(a \pm b)^2+t^2.$$

As the hypothenuse, c, must be greater than either of the sides of the triangle CPZ, the following conclusions may be deduced from the above equation:

When, as in figure 95, c is less than $(a-b)$, the circles lie

within the other, and cannot be connected by a common tangent.

When, as in figure 96, $c=(a-b)$, the curves touch on the inside and are compound.

When, as in figure 97, c is less than $(a+b)$ and greater than $(a-b)$, the curves intersect; there will be one pair of common tangents, and the curves will be compound, that is to say, the centres will lie on the same side of the common tangent.

When, as in figure 98, c is equal to $(a+b)$, the curves touch externally at a point of reversed curvature, and there is also a pair of tangents to which the curves are compound.

When, as in figure 99, c is greater than $(a+b)$, there are two pairs of tangents, and the curves are reversed to one and compound to the other.

In each case the line **CZ** will pass through the intersection of the pair of tangents, and the angles between **CZ** and the radii from either centre to the extremities of the tangents will be equal and on opposite sides of the line connecting the centres.

2. *Formulæ.* Measure the distance, $(\mathbf{AB})=i$, between any two convenient points, and also the greater and lesser acute angles, A and B′, between **AB** and the tangents through A and B; note also whether each curve is convex or concave to **AB**, and whether the centres are on the same or different sides of that line.

The origin of co-ordinates should, for convenience, be the centre of the curve with the longer radius. In this problem, it is assumed to be at the centre of the curve through A with radius a.

When at the centre of the curve through B, change, in all the formulæ, B to A, and A to B. Looking from C towards **AB**, the right will be positive, with reference to bearings and departures, when B is to the right of A, and negative when B is to the left of A. As more fully explained in article 5, proposition XVI., the axis **CN** is perpendicular to **AB**; A, C, and B are the bearings of CA, CZ, and BZ; $-T$ and $+T$ are the deflections from CZ to the radii from C to the extremities, Ta and Ta', of the pair of common tangents; $(C \pm T)$ is the bearing of those radii; la and lb are the tangential deflections from A to one extremity, Ta, of the common tangent, and from the other extremity, Tb, to B. Σd and Σl are the alge-

braic sums of the departures and latitudes of (**CA**)=a, (**AB**) =i, and (**BZ**)=b, equal to the departure and latitude of the closing line (**CZ**)=c.

$\Sigma d = +i \pm a.\sin.\ A' \pm b.\sin.\ B'$. (1)

$\Sigma l = +a.\cos.\ A' \pm b.\cos.\ B'$. (2)

Tan. $C = \Sigma d \div \Sigma l$. (3)

Cos. $T = (a \pm b).\sin.\ C \div \Sigma d = (a \pm b).\cos.\ C \div \Sigma l$. (4)

$t = \Sigma d.\sin.\ T \div \sin.\ C = \Sigma l.\sin.\ T \div \cos.\ C = (a \pm b).\tan.\ T$. (5)

$A = A'$, and is always acute. (6)

$$B = \begin{cases} B' & \text{for centres on different sides of } \mathbf{AB}, \\ 180° - B' & \text{" " the same side "} \end{cases} \quad (7)$$

$$A \text{ or } B \text{ is } \left\{ \begin{array}{l} \text{positive when the curve through A or} \\ \quad \text{B is convex} \\ \text{negative when the curve through A or} \\ \quad \text{B is concave} \end{array} \right\} \text{to } \mathbf{AB}.$$

$Ia = (C \pm T) - A$. (8)

$$Ib = \begin{cases} B - (C \pm T) & \text{when the curves are reversed.} \\ B - (C \pm T) + 180° & \text{" " " " compound.} \end{cases} \quad (9)$$

(1) $a.\sin.\ A'$, or $b.\sin.\ B'$, is positive when the curve through A or B is convex, and negative when it is concave.

(2) $b.\cos.\ B'$ is positive when the centres are on different sides of **AB**, and negative when they are on the same side.

(3) The sign of C is the same with the sign of Σd. If C′ is the acute angle corresponding to the numerical value of the tangent, C is equal to C′ when Σl is positive, and to 180°−C′ when Σl is negative.

(4) Substitute $(a+b)$ for $(a \pm b)$ when the curves are reversed, and $(a-b)$ when they are compound. If the resulting value of cos. T is greater than unity, the solution is not possible for the given values of the radii. T is acute and equal to T′ for reversed curves, and also for compound curves when a is greater than b and $(a-b)$ is therefore positive; but when the curves are compound and $(a-b)$ is negative, $T = 180° - T'$. This is on the assumption that the origin, C, is at the centre of the curve through A. If at the centre of the curve through B, T would be obtuse should $(b-a)$ be negative. For this reason, in order that T may be acute, it is best to assume C to be at the centre of the curve with the longer radius.

In (5), the last formula, $t = (a \pm b).\tan.\ T$, is the most con-

venient for general use, substituting $(a+b)$ for $(a\pm b)$ when the curves are reversed, and $(a-b)$ when they are compound.

(7), (8), and (9). Note the distinction between A′ and A, or between B′ and B, the former indicating an acute angle, and the latter an angle which may be acute or obtuse according to the value derived from the formula. As the differences are algebraic, care must be exercised to give to each angle its proper sign, and when the sign is negative the sign prefixed in the formula must be changed from plus to minus, or from minus to plus. When the sign of Ia or Ib is negative, it indicates that the tangential deflection over the curve is negative. The algebraic sum, $Ia+Ib$, should be equal to the angle, I, of intersection, computed in terms of A′ and B′ by the rules of problem V. There will always be two values of Ia and Ib corresponding to the two positions of the pair of equal tangents, and one of these values will be obtained by substituting (C−T) for (C ± T), and the other by substituting (C+T), in the formula.

If these rules are observed, (8) and (9) will give the signs and magnitudes of Ia and Ib for both positions of the tangents, and for the curves whether compound or reversed. If the calculated values of these angles are greater than 180°, subtract from 360° and change the sign.

In table XII., the values of Σd, Σl, C, Ia, and Ib are given for the different figures to which reference is made.

3. *Example* 1 (figure 114). $i=(\mathbf{AB})=1212.6$; A′=34° 33′; B′=23° 30′; $a=955.4$; $b=573.6$; both curves are convex, and the centres are on the same side of **AB**, so that I is equal to (A′+B′) when the curves are compound, and to 180°−(A′+B′) when they are reversed, according to the rules of problem V.

(1) $\Sigma d = +i+a$. sin. A′$+b$.sin. B′ = 1212.6+541.7+228.7= +1983.0.

(2) $\Sigma l=+a$.cos. A′$-b$.cos. B′=787.0−526.1=+260.9.

(3) Tan. (C=+82° 30′)=$\Sigma d\div\Sigma l$=(+1983.0)÷(+260.9)=.7602.

If the curves are to be compound:

(4) Cos. (T=79° 00′)=$(a-b)$.sin. C÷Σd=381.7.sin. C+1983.0.

(5) $t=(a-b)$.tan. T=381.7.tan. 79° 00′=1963.5.

(6) and (7) A=+A′=+34° 33′. B=+(180°−B′)=+156° 30′.

(8) Ia=(C−T)−A=−31° 03′. I′a=(C+T)−A=−126° 57′.

(9) Ib=B−(C−T)−180°=−27° 00′. I′b=B−(C+T)+180°= +175° 00′.

Note that the algebraic difference $I'a - Ia = 158° = 2\ T$; the algebraic difference $I'b - Ib = 203°$, which is equivalent to 158°; $I = Ia + Ib = A' + B' = -58°\ 03'$. $I'a + I'b = +126°\ 57' + 175°\ 00' = +301°\ 57'$, which is an equivalent deflection to —58° 03', one being to the right and the other to the left.

When the curves are reversed, substitute in (4) and the succeeding formulæ $(a+b)$ for $(a-b)$.

(4) Cos. $(T = 40°\ 08') = (a+b).\sin.\ C + \Sigma d = 1529.\sin.\ 82°\ 30' + 1983.0$.
(5) $t = (a+b).\tan.\ T = 1529.\tan.\ 40°\ 08' = 1289.0$.
(6) and (7) $A = +34°\ 33'$. $B = (180° - 23°\ 30') = +156°\ 30'$.
(8) $Ia = (C - T) - A = +7°\ 49'$. $I'a = (C + T) - A = +88°\ 05'$.
(9) $Ib = B - (C - T) = +114°\ 08'$. $I'b = B - (C + T) = +33°\ 52'$.

In this case, the curves are reversed and the centres are on the same side of **AB**, and therefore $I = 180° - (A' + B') = 121°\ 57' = Ia + Ib$, and the algebraic difference $Ia - Ib = 2\ T = 80°\ 16'$.

In the calculation of Ia, Ib, $I'a$, and $I'b$, the rules for the magnitudes and signs of the deflections C, T, A, and B should be carefully observed, and the plus and minus signs understood in the algebraic sense.

Example 2 (figure 101). In problem XXIII., example 2, a pair of reversed curves with equal radii are calculated for an intermediate tangent of 300 feet. In this example, the same radii are given, and it is required to determine the intermediate tangent.

$i = 600$; $A' = 18°\ 00'$; $B' = 6°\ 00'$; $a = b = 520$; the curves are reversed, with centres on different sides of **AB**, and both concave.

(1) $\Sigma d = i - a.\sin.\ A' - b.\sin.\ B' = +384.7$.
(2) $\Sigma l = a.\cos.\ A' + b.\cos.\ B' = +1011.8$.
(3) Tan. $(C = +20°\ 49') = \Sigma d \div \Sigma l = .3802$.
(4) Cos. $(T = 16°\ 06') = (a+b).\sin.\ C \div \Sigma d = .9608$.
(5) $t = (a+b).\tan.\ T = 300.1$.
(6) and (7) $A = -A' = -18°\ 00'$. $B = -B' = -6°\ 00'$.
(8) and (9) $Ia = (C - T) - (-A') = +22°\ 43'$. $Ib = (-B') - (C - T) = -10°\ 43'$.

As the curves are reversed, with centres on different sides of

AB′, and the triangle AIB is similar to figure 7a, I=(A′—B′) =12° 00′, which is equal to the algebraic sum of I*a* and I*b*. The values of these angles for the other tangent of the pair is not calculated.

PROBLEM XXVI. (TABLE XII.)

(*For demonstration see proposition XVI., article* 6.)

In problem XXV., the radii and positions of two curves are given, to determine the lengths and positions of the pair, or pairs, of connecting tangents.

In this problem the length, t, of the common tangent, and one of radii, are given, to determine the other radius.

The first step is to determine the turn, or deflection, I, by the rules of problem V. It will be equal to (A′ ± B′) or to 180°—(A′ ± B′), and the values are given in table XII., for different figures, with the equations for the solution of the problem for the particular form of curves shown in the figure.

a or *b* is the given radius; *b* or *a* the required radius; (**AB**) = *i* is the distance from A to B; A′ and B′ are the greater and lesser acute angles between **AB** and the tangents through A and B; *t* is the length of the common tangent to the two curves; it is to be noted whether the curves are concave or convex to **AB**, whether they are compound or reversed, and whether the centres are on the same or different sides of **AB**.

(2) *Formulæ*. Either of the formulæ below may be used at pleasure. (2) is more convenient for tabulation, and is used in table XII., in which the signs of the terms are given for the forms of curves in the figures to which reference is made. (1) will be found more convenient for general use. Note that *a* is the radius corresponding to the greater of the two angles A′ and B′.

$$a=\frac{\frac{1}{2}(i^2-t^2)\div i\pm b.\text{sin. B}'}{\mp\text{sin. A}'\pm b\,.\,\text{vers. I}\div i} \qquad b=\frac{\frac{1}{2}(i^2-t^2)\div i\pm a.\text{sin. A}'}{\mp\text{sin. B}'\pm a\,.\,\text{vers. I}\div i} \quad (1)$$

$$a=\frac{(+m\pm b).p}{(\mp n\pm b)}. \qquad b=\frac{(+o\pm a).n}{(\mp p\pm a)}. \quad (2)$$

In (2), m, n, o, and p have the following values:

$m = \frac{1}{2}(i^2 - t^2) \div (i \cdot \sin. B')$. $\quad o = \frac{1}{2}(i^2 - t^2) \div (i \cdot \sin. A')$.

$n = i \cdot \sin. A' \div \text{vers. I.}$ $\quad p = i \cdot \sin. B' \div \text{vers. I.}$

The term $\frac{1}{2}(i^2 - t^2) \div i$ may, for convenience of calculation, be placed under the form $\frac{1}{2}(i+t) \times (i-t) \div i$.

The signs of the terms in (2), and also of the corresponding terms in (1), are determined by the following rules:

m has always a plus sign prefixed, but becomes negative when t is greater than i.

b, in the numerator, is plus for convex, and minus for concave, curves.

n is plus when the curve through A is concave, and minus when it is convex.

b, in the denominator, is plus when the curves are reversed, and minus when they are compound.

The signs in (2b) are determined in the same manner.

o has always a plus sign prefixed, but becomes negative when t is greater than i.

a, in the numerator, is plus for convex, and minus for concave, curves.

p is plus when the curve through B is concave, and minus when it is convex.

a, in the denominator, is plus when the curves are reversed, and minus when they are compound.

N.B. The solution for any proposed form of curves is not possible for any value of a radius which, when substituted in the second member of the equations, gives unlike signs to the numerator and denominator, and thereby renders the radius in the first member negative.

The equations for different forms of curves, and the limits of the radii within which the solution is possible, are given in table XII. When the second radius is calculated, Ia and Ib are determined by problem XXV., and these equations are also given in the table.

(3) *Examples.* It will be seen by reference to the figures that there are two classes of curves. In one, the common tangent is on the alignment, as in figures 101 and 102. In the other, as in figures 105 and 107, the alignment is to be retraced to a tangent-point, Ta or Tb, and another curve substituted. In the first case, equal radii may be calculated by problem

XXIII., or radii of given values calculated by problem XXV., or one radius may be assumed and the other calculated by this problem. In the second case, the curve must be retraced with the radius with which it has been run, and if it does not contain the calculated deflection, I*a* or I*b*, the solution is not possible.

Example 1 (figure 102). $i = (\mathbf{AB}) = 1300$; $A' = 30° 30'$; $B' = 4° 00'$; curve through A concave, curve through B convex; the curves are reversed, with centres on different sides of **AB**; the triangle AIB is similar to figure 7, and therefore $I = (A' + B') = 34° 30'$.

The degree of curve through B is 3° 30′; radius $b = 1637.3$, and the length of the intermediate tangent is $t = 300$ feet. Required the radius a.

To obtain the value of a, it is necessary to calculate only m, n, and p, but o may be determined with but little additional calculation, and, with this value of v, the computations may be checked in the manner shown in the example.

Computation of m and o:

$\frac{1}{2}(i+t) = 800$...	log.	2.903090			
$(i-t) = 1000$....	"	3.000000			
$i = 1300$.....a.c.	"	6.886057			
$\frac{1}{2}(i^2-t^2)+i$....	"	2.789147			2.789147
Sin. (B=4° 00′).	"	8.843585	sin.(A=30° 30′)	log.	9.705469
$m = 8821.9$.....	"	3.945562	$o = 1212.5$	"	3.083678

Computation of n and p:

$i = 1300$.....log.		3.113943			
Vers. (I=34° 30′)...	"	9.245202			
i+vers. I.....	"	3.868741			3.868741
Sin. (A=30° 30′)....	"	9.705469	sin. (B=4° 00′)	log.	8.843585
$n = 3751.5$.....	"	3.574210	$p = 515.6$	"	2.712326

Calculation of a:

$m+b = 10459.2$.....	log.	4.019499
$p = 515.6$.....	"	2.712326
$n+b = 5388.8$.....a.c.	"	6.268597
$a = 1000.8$.....	"	3.000332

This last equation, determined by the rules above, is given n table XII., opposite figure 102. Having the value of o, the

work may be checked by computing $b=(o-a).n\div(a-p)$. The solution would not be possible for any value of a greater than o or less than p, because such a value would result in unlike signs for the numerator and denominator, and a negative value for b.

When both radii are known, the tangential deflections, Ia and Ib, are determined by problem XXV. and according to the formulæ of table XII. These formulæ for the curves represented in the figure under consideration are as follows:

$\Sigma d=+i-a.\sin.\ A+b.\sin.\ B=1300-508.0+114.2=+906.2.$
$\Sigma l=a.\cos.\ A'+b.\cos.\ B'=862.3+1633.3=+2495.6.$
$\text{Tan.}\ (C=+19°\ 57')=\Sigma d\div\Sigma l=+906.2\div(+2495.6)=0.36311.$
$\text{Tan.}\ (T=6°\ 29')=t\div(a+b)=300\div2638.1=0.11371.$
$Ia=(C-T)-(-A')=43°\ 58'.\quad Ib=+B'-(C-T)=-9°\ 28'.$

The deflection from A to Ta is therefore 43° 58′ to the right, and the deflection from Tb to B 9° 28′ to the left, and the algebraic sum $Ia+(-Ib)=34°\ 30'=(A'+B')=I$.

By adding and subtracting (C+T), the deflections, I′a and I′b, to the second of the pair of common tangents would be obtained, but this tangent would not be available for an alignment of a railroad-track.

Example 2 (figure 104). $i=600$; $A'=18°\ 00'$; $B'=6°\ 00'$; $t=300$; $a=520.0$; curves reversed, with centres on different sides of **AB**; the triangle AIB is similar to figure 7a, and therefore, by problem V., the deflection over the curves is $I=(A'-B')=(12°\ 00')$. The equations for this form of curves are those opposite figure 104 in table XII.

In this case the calculation will be made by formula (1b), the signs of the terms being the same as in the table.

Calculation of $\frac{1}{2}(i^2-t^2)\div i$; $a.\sin.\ A'$; and $a.\text{vers.}\ I\div i$:

$\frac{1}{2}(i+t)=450$.....	log. 2.653213	$a=520$	log. 2.716003
$(i-t)=300$.....	" 2.477121	$\sin.(A=18°\ 00')$	" 9.489982
$i=600$..... a.c.	" 7.221849	$a.\sin.\ A=160.7$	log. 2.205985
$\frac{1}{2}(i^2-t^2)\div i=225.0$.	" 2.352183		
$a=520$.....	" 2.716003		
Vers. (I=12° 00′)...	" 8.339499		
$i=600$..... a.c.	" 7.221849		
$a.\text{vers.}\ I\div i=.01894$.	" 8.277351		

Calculation of b:

$\frac{1}{2}(i^2-t^2)\div i-a.\sin.\ A'=64.3$ ..... log. 1.808211
Sin. $(B=6°\ 00')+a.\text{vers. } I\div i=.1235$ ..... " 9.091667
$b=520.6$ ..... " 2.716544

This is the same with example 2, problem XXIII., in which a common tangent of 300 feet was given, to determine a pair of curves with equal radii. The difference of six-tenths in the value of the radii, in this example, is due to the neglect of hundredths in the computations.

Calculation of I*a and* I*b:*

$\Sigma d=+i-a.\sin.\ A'-b.\sin.\ B'=600-160.7-54.4=+384.9.$
$\Sigma l=a.\cos.\ A'+b.\cos.\ B'=494.6+517.8=+1012.4.$
Tan. $(C=+20°\ 49')=\Sigma d\div\Sigma l=+384.9\div(+1012.4)=.3801.$
Tan. $(T=16°\ 05')=t\div(a+b)=300\div1040.6=.2883.$
$Ia=(C-T)-(-A')=+22°\ 44'.$ $Ib=(-B')-(C-T)=(-6°)-4°\ 44'=-10°\ 44'.$
$Ia+(-Ib)=+12°\ 00'=(A'-B')=I.$

Example 3 (figure 107). A 7-degree curve has been run to a point B, with a tangential direction IB, and it is required to change the alignment by retracing the curve to a point T*b*, and reversing with an intermediate tangent of 300 feet, so that the curve shall terminate at a point A with a tangential direction IA.

The angles, A′=BAI, and B′=ABI, are 70° and 36°; the triangle AIB is similar to figure 7; the curves are reversed, with centres on the same side of **AB**, and therefore, by problem V., the algebraic sum, $Ia+Ib=I$, of the deflections over the curves from B to A, will be equal to $180°-(A'+B')=74°\ 00'$.

$i=(\mathbf{AB})=400$; $b=819.0$; $t=300$; both curves convex to **AB**; required *a*, I*a*, and I*b*.

For the equations, see table XII., figure 107.

$a=(m+b).p\div(b-n)=(148.9+819.0)\times324.6\div(819.0-519.0)=1047.2.$
$\Sigma d=i+a.\sin.\ A'+b.\sin.\ B'=400+984.1+481.4=+1865.5.$
$\Sigma l=a.\cos.\ A-b.\cos.\ B=358.2-662.6=-304.4.$
Tan. $(C=+99°\ 16')=\Sigma d\div\Sigma l=+1865.5\div(-304.4).$
Tan. $(T=9°\ 08')=t\div(a+b)=300\div1866.2.$

Observe that C is obtuse because Σl is negative.

Problem XXV. (6) and (7). $A = (+A') = +70°$. $B = +(180° - B') = +144°$, because the centres are on the same side of **AB**.

(8) $Ia = (C - T) - A = +20°\ 08'$. (9) $Ib = B - (C - T) = +53°\ 52'$.

Ia and Ib may also be calculated by the formulæ of table XII., in which they are given in terms of the acute angles, A', B', $C' = 80°\ 44'$, and $T = 9°\ 08'$. $Ia = 180° - (C' + T) - A' = 20°\ 08'$. $Ib = -B' + (C' + T) = +53°\ 52'$. A small difference in the value of the radii will, however, often change a bearing from acute to obtuse, or from positive to negative, and it is safest to use the formulæ of problem XXV. Ia, when C is the centre of the curve through A, or Ib when it is the centre of the curve through B, may be easily determined, and, one being known, the other is obtained from the equation $Ia + Ib = I$. In this example, for instance, $Ia = 20°\ 08'$, $I = 74°\ 00'$, and $Ib = I - Ia = 53°\ 52'$.

Example 4 (figures 88 and 115). In the preceding example, referring to figure 107, the radius b was given, and the radius a determined by the formula, $a = (m + b).p \div (n - b)$.

If, in this equation, $b = n$, a would be infinite, the curve through A would coincide with the tangent **AI**, and the position of the point of contact with the curve through B would be determined by problem XIX. For a value of b less than n the denominator would be negative; the direction of the radius a would be reversed; the curve through A would be, therefore, concave, and the curves would be compound, as in figures 115 and 88.

When the curves are compound, the point of contact should be a point and not a line, as in figure 88, in which the curves are the same as in figure 115, except that t is equal to zero.

In figure 107, the curves are reversed, with centres on the same side of **AB**, and therefore, by problem V., $I = 180° - (A' + B')$.

In figure 88, the radius through A is reversed in direction and the centres are on different sides of **AB**; but as the curves become compound, the value of I remains the same. In figure 107, $Ia + Ib = I$, while in figure 88, $Ia - Ib = I$.

Suppose the curves to be as in figure 88; $b = 478.3$, $t = 0$, and the remaining lines and angles as in the last example.

$i = 400$; $A' = 70°$; $B' = 36°$; $I = 180° - (A' + B')$; the curve through A concave, and the curve through B convex. Required a, Ia, and Ib.

The equations for this figure are not given in table XII., but may be obtained as follows by making $t = 0$ in (1a) of this problem:

$$a = (\tfrac{1}{2} i + b.\sin.\ B') \div (+\sin.\ A' - b.\text{vers.}\ I + i).$$

In this equation, $\frac{1}{2} i$ is always positive; $b.\sin.\ B'$ is positive because the curve through B is convex; $\sin.\ A'$ is positive because the curve through A is concave, and the sign of $b.\text{vers.}\ I \div i$ is negative because the curves are compound. Observe that the second term of the numerator is positive for convex curves, and the first term of the denominator positive for concave curves.

Substituting the values of the terms in this equation,

$$a = (200 + 281.1) \div (+.93969 - .86623) = 6536.3.$$

The calculation may be checked by computing the value of b from this value of a by the following equation:

$$b = (\tfrac{1}{2} i - a.\sin.\ A') \div (-\sin.\ B' - a.\text{vers.}\ I + i) = 478.3.$$

The signs in this equation are determined by the same rules as in the calculation of a. The second term of the numerator is negative because the curve through A is concave; the first term of the denominator is negative because the curve through B is convex, and the second term of the denominator is negative because the curves are compound. As both terms of the denominator are negative, the solution is possible only when the numerator is negative, or when $a.\sin.\ A'$ is greater than $\frac{1}{2} i$. When $\frac{1}{2} i = a.\sin.\ A'$, $b = 0$, the curve through A passes also through B, and $i = 2\, a.\sin.\ A'$ is the chord.

Calculation of Ia *and* Ib *by the formulæ of problem XXV.:*

(1) $\Sigma d = +i - a.\sin.\ A' + b.\sin.\ B' = +400 - 6142.1 + 281.1 = -5461.0.$

(2) $\Sigma l = a.\cos.\ A' + b.\cos.\ B' = 2235.5 + 387.0 = +2622.5.$

(3) $\text{Tan.}\ (C = -64°\ 21') = \Sigma d \div \Sigma l = -5461.0 \div (+2622.5).$

As $t = 0$, $T = 0$.

(6) and (7) $A=(-A')=-70°$. $B=(+B')=+36°$.
(8) and (9) $Ia=C-A=+5°\ 39'$. $Ib=B-C-180°=-79°\ 39'$.
$Ia+(-Ib)=-79°\ 39'-(+5°\ 39')=-74°=I=180°-(A'+B')$.

As the degrees of the curves through A and B are ($0°\ 52'=0.87°$) and $12°\ 00'$, the required change of alignment from B to A is made by turning 6.63 stations to the left from B to T, and thence 6.49 stations to the right from T to A.

The same results, by a different method, would be obtained from the equations of table X. for figure 88. Those equations, however, apply only when, as in the case under consideration, $t=0$.

Example 5 (figure 101). *Reversed curves between parallel tangents.*

In this case, $A'=B'$; $I=A'-B'=0$; both curves are concave to **AB**, and the equations are as follows:

(1a) $a=[\frac{1}{2}(i^2-t^2)\div i-b.\sin.\ B']\div\sin.\ A'=\frac{1}{2}(i^2-t^2)\div i.\sin.\ A'-b.$

$b=[\frac{1}{2}(i^2-t^2)\div i-a.\sin.\ A']\div\sin.\ B'=\frac{1}{2}(i^2-t^2)\div i.\sin.\ B'-a.$

If $i=1300$, $t=300$, $a=1480$, $A'=B'=12°\ 00'$, then b will be equal to 1480. It does not, of course, follow that the radii will be equal in every case, but this is the same with example 3, problem XXIII., of a pair of reversed curves with equal radii and an intermediate tangent of 300 feet.

Calculation of Ia *and* Ib *by the formulæ of problem XXV.:*

(1) $\Sigma d=i-2\,a.\sin.\ A'=1300-615.4=+684.6$.
(2) $\Sigma l=2\,a.\cos.\ A'=+2895.1$.
(3) Tan. $(C=+13°\ 18')=\Sigma d\div\Sigma l=+684.6\div(+2895.1)$.
(5) Tan. $(T=5°\ 47')=\frac{1}{2}\,t\div a$.
(6) and (7) $A=B=(-A')=-12°\ 00'$.
(8) and (9) $Ia=Ib=(C-T)-(-A)=+19°\ 31'$.

4. From the formula, $a=(m-b).p\div(n-b)$, may be derived all of the formulæ of the preceding problems with relation to curves, compound or reversed, or intersecting as in the case of sidings, crossings, and turn-outs; in problem XIX., for instance, by making $b=n$, and a, therefore, infinite; in problem

XX., by making $b=m$, and a, therefore, equal to zero; in problem XXII., by making t equal to zero; in problem XXIII., by making a equal to b; and in problem XXVII., by making t equal to zero, and substituting in the denominator vers. I— vers. F for vers. I, as explained in the demonstrations of those problems in proposition XVI., article 11.

These formulæ, as well as those of problem XXV., are derived from the rules for departures and latitudes in problem XI., and for algebraic sums and differences of deflections in problem III., and present no difficulties, provided that care is taken to note the sign of each bearing and whether it is acute or obtuse, in accordance with the rules given, which are in every case without exceptions and readily applied.

PROBLEM XXVII.

(*For demonstration see proposition XVI., article* 10.)

A turn, or deflection, is to be made over two curves from one given tangential direction to another, and one of the radii, as a, and the deflection, I*a, over the curve through* **A**, *are given; required* I*b, b, and the length, t, of the intermediate tangent to the two curves.*

Measure the distance (**AB**) = i between the tangent-points A and B, and the greater and lesser acute angles, A′ and B′, between **AB** and the tangents through A and B, and calculate the angle, I, of intersection by the rules of problem V.

1. *Solution by construction* (figure 100). Through **A** and **B** with a radius $\frac{1}{2}\,i \div \sin.\ \frac{1}{2}$ I, describe a circle AKB.

When t is a point, the points of compound and reversed curvature will be on this circle, as in figure 78, problem XXII.

When t is not a point, the *chords of the arcs* will intersect on the circle, and therefore the solution will be as follows:

From the centre C, with the given radius, a, describe an arc ATa of Ia degrees, and from Ta produce the tangent in the direction Tb.

From A draw the chord **AT***a* to intersection with the circle AKB at K. From B through K draw a line cutting the tangent from T*a* at a point T*b*. (**T***b***T***a*) = *t* will be the required intermediate tangent, and **BT***b* the chord of the curve through B.

From the middle of this chord draw a perpendicular to intersection with the radial line through B at a point Z. (**ZB**)=*b* is the radius of the arc through B, and the angle BZT*b* is equal to the tangential deflection, I*b*, over the arc.

Demonstration. By problem V., the algebraic sum, I*a*+I*b*, of the tangential deflections over the arcs is equal to the angle, I, of intersection, and by problem VII. the deflection, K, between the chords is equal to ½ I whether *t* is a point or line.

If, therefore, through A and B, with a radius ½ *i* ÷ sin. ½ I, the circle AKB be drawn, the central angle between the radii terminating at A and B will be equal to I, and the angle between each pair of chords, from A and B, intersecting on the circle, will be equal to ½ I, and conversely every pair of chords intersecting at an angle ½ I will cross on the circumference of the circle AKB.

2. *Formulæ.* *a* and *b* are the radii; *i* = (**AB**) is the distance from A to B; *t* = (**T***a***T***b*) is the intermediate tangent; A′ and B′ are the acute angles between **AB** and the tangents through A and B; I*a* and I*b* the tangential deflections over the arcs AT*a* and T*b*B; I=I*a*+I*b* the algebraic sum of the tangential deflections, determined by the rules of problem V. in terms of A′ and B′, and (C ± T) the symbol used in proposition XXV. to represent the bearings from an axis **CN**, at right angles to **AB**, to the radial lines (**CT***a*) and (**CT***b*), equal to the angle between **AB** and the tangent (**T***a***T***b*)=*t*.

Problem V. I*a*+I*b*=I. (1)

Problem XXV. (8) and (9) (C ± T)=I*a*+A=B+I*b*. (2)
a vers. I*a*−*b*. vers. I*b*=*i*. sin. (C ± T). (3)
t=*a*. sin. I*a*+*b*. sin. I*b*+*i*. cos. (C ± T). (4)

The signs of the terms in these equations depend upon the signs and magnitudes of the angles, and are given, for different forms of curves, in the following table:

Figure	Curves	I	(1)		(2)	(3)				(4)		
			Ia	Ib	(C±T)	a		b		t		
102	Rev'd	A′+B′=Ia−Ib	I+Ib	Ia−I	Ia−A′=B′+Ib	+i.sin. (C±T)−b.vs. Ib vs. Ia		+i.sin. (C±T)−a.vs. Ia vs. Ib		+i.cos. (C±T)−a.sin. Ia−b.sin. Ib		
104	"	A′−B′=Ia−Ib	I+Ib	Ia−I	Ia−A′=Ib−B′	+	−	+	−	+	−	−
105	"	A′−B′=Ia+Ib	I−Ib	I−Ia	A′−Ia=Ib+B′	+	−	+	−	−	−	+
107	"	180°−(A′+B′)=Ia+Ib	I−Ib	I−Ia	Ia+A′=180°−(Ib+B′)	+	−	+	−	−	−	+
111	Comp'd	A′+B′=Ia+Ib	I−Ib	I−Ia	A′−Ia=Ib−B′	−	+	+	+	+	−	−
113	"	A′−B′=Ia−Ib	I+Ib	Ia−I	A′−Ia=B′−Ib	+	+	−	+	+	+	−
113½	"	A′−B′=Ia−Ib	I+Ib	Ia−I	A′−Ia=B′−Ib	−	+	+	+	+	−	+
113½	"	A′−B′=Ia−Ib	I+Ib	Ia−I	Ia−A′=Ib−B′	+	+	−	+	+	+	−
115	"	180°−(A′+B′)=Ia−Ib	I+Ib	Ia−I	180°−(A′+Ia)=B′−Ib	+	+	−	+	+	+	−
115½	"	180°−(A′+B′)=Ib−Ia	Ib−I	Ia+I	A′−Ia=180°−(B′+Ib)	+	+	−	+	+	+	−
116	"	180°−(A′+B′)=Ib−Ia	Ib−I	Ia+I	A′−Ia=180°−(B′+Ib)	−	+	+	+	+	−	+
116½	"	180°−(A′+B′)=Ia−Ib	Ib+I	Ia−I	180°−(A′+Ia)=Ib−B′	−	+	+	+	+	−	+
117	"	I=Ib−Ia . i=0	Ib−I	Ia+I	i=0	b.vers. Ib+vers. Ia		a.vers. Ia+vers. Ib		a.sin. Ia−b.sin. Ib		
118	"	I=0=Ib−Ia	Ib	Ia	90°−Ia	b+i.cos. Ia+vers. Ia.		a−cos. Ia+vers. Ia		$\begin{cases} (i+a-b) \text{ . sin. } Ib \\ =(a-b) \text{ . tan. } Ia \end{cases}$		
119	"	I=0=Ib−Ia	Ib	Ia	Ia=Ib	a=b		b=a		t=p+sin. Ib		

The plus and minus signs in this table are to be understood in the arithmetic sense, and the terms are so arranged as to give positive results. A negative result will show that the solution is not possible for the curves shown in the figure to which reference is made.

3. *Examples.* In the application of the formulæ of this problem, there are two classes of curves. In one, represented in figures 102, 104, and 111, both curves and the intermediate tangent are a portion of the alignment. In the other class, represented in figures 105, 107, 108, 113, and 115, one of the arcs, as TaA, has been run to a point A, from a point Ta on a tangent-line, and it is required to substitute a second curve, which, starting from a point Tb on the same tangent-line, will pass through a point B with a given tangential direction.

Example 1 (figure 102). *Reversed Curves.*—$i = (\mathbf{AB}) = 1300$; $A' = 30° 30'$; $B' = 4° 00'$; $I = (A' + B') = 34° 30'$; $b = 1637.3$, and $Ib = 9° 28'$.

Required the deflection, Ia, over the curve through A; the radius, a, and the length, t, of the intermediate tangent. See example 1, problem XXVI.

(1) $Ia = I + Ib = 43° 58'$.

(2) $(C \pm T) = Ia - A' = Ib + B' = 13° 28'$.

(3) $a = \left(\begin{matrix} +i.\ \sin.\ (C \pm T) & -b.\ \text{vers.}\ Ib \\ 302.8 & 22.3 \end{matrix}\right) \div \left\{\begin{matrix} \text{vers.}\ Ia \\ \text{vers.}\ 43° 58' \end{matrix}\right\} = 1000.8.$

(4) $t = \left(\begin{matrix} i.\ \cos.\ (C \pm T) & -a.\ \sin.\ Ia & -b.\ \sin.\ b \\ 1264.2 & 694.8 & 269.3 \end{matrix}\right) = 300.1.$

Example 2 (figure 107). *Reversed Curves.*—$i = 400$; $A' = 70°$; $B' = 36°$; $I = 180° - (A' + B') = 74° 00'$; $b = 819.0$; $Ib = 53° 52'$. (See example 3, problem XXVI.)

(1) $Ia = I - Ib = 74° - 53° 52' = 20° 08'$.

(2) $(C \pm T) = Ia + A' = 180° - (Ib + B') = 90° 08'$.

(3) $a = \left(\begin{matrix} i.\ \sin.\ (C \pm T) & -b.\ \text{vers.}\ Ib \\ 400.0 & 336.1 \end{matrix}\right) \div \left\{\begin{matrix} \text{vers.}\ Ia \\ \text{vers.}\ 20° 08' \end{matrix}\right\} = 1046.0.$

(4) $t = \left\{\begin{matrix} -i.\ \cos.\ (C \pm T) & -a.\ \sin.\ Ia & +b.\ \sin.\ Ib \\ 0.93 & 360.04 & 661.4 \end{matrix}\right\} = 300.4.$

Example 3 (figure 111). *Compound Curves.*—$i = 950$; $A' = 8° 00'$; $B' = 7° 00'$; $I = (A' + B') = 15°$; $a = 3000$; $Ia = 5° 42'$. (See example 4, problem XXII.)

(1) $Ib = I - Ia = 15° - 5° 42' = 9° 18'.$

(2) $(C \pm T) = A' - Ia = Ib - B' = 2° 18'.$

(3) $b = \left(\begin{matrix} i.\ \text{sin. } (C \pm T) & + & a.\ \text{vers. } Ia \\ 38.13 & & 14.83 \end{matrix}\right) \div \begin{matrix} \text{vers. } Ib \\ \text{vers. } 9.18' \end{matrix} = 4029.$

(4) $t = \left(\begin{matrix} i.\ \text{cos. } (C \pm T) & - & a.\ \text{sin. } Ia & - & b.\ \text{sin. } Ib \\ 949.2 & & 298.0 & & 651.1 \end{matrix}\right) = 0.1.$

Example 4 (figure 113). *Compound Curves.*—Three stations of a seven-degree curve have been run to a point A from a point Ta on a tangent-line. It is required to shift the tangent-point to a point Ib on the same line, so that the curve shall pass through a point B with a tangential direction IB.

i=100 is the distance from A to B; A'= 30° 00'; B'=26° 00'; I=(A'−B')=4° 00' is the change of tangential direction; Ia= 21° 00'; a=819.0. Required the tangential deflection, Ib, from Tb to B; the distance, t, from Ta to Tb, and the radius, b, of the arc TbB.

(1) $Ib = Ia - I = 17° 00'.$

(2) $(C \pm T) = A' - Ia = B' - Ib = 9° 00'.$

(3) $b = \left(\begin{matrix} -i.\ \text{sin. } (C \pm T) & + & a.\ \text{sin. } Ia \\ 15.64 & & 54.40 \end{matrix}\right) \div \begin{matrix} \text{vers. } Ib \\ \text{vers. } 17° \end{matrix} = 887.0.$

(4) $t = \left(\begin{matrix} i.\ \text{cos. } (C \pm T) & + & a.\ \text{sin. } Ia & - & b.\ \text{sin. } Ib \\ 98.77 & & 293.26 & & 259.35 \end{matrix}\right) = 132.7.$

Tb is therefore 132.7 feet forward from Ta; the degree, Db, of the curve through B is 6° 28'= 6.47°; and the number of stations from Tb to B is 17.00°÷ 6.47°= 2.63.

Example 5 (figure 113¼). The angles and lines are the same as in the last example, but the directions of both radii are reversed, so that the curve through A becomes concave, and the curve through B convex, to **AB.**

(1) $Ib = Ia - I = 21° 00' - 4° 00' = 17° 00'$

(2) $(C \pm T) = A' - Ia = B' - Ib = 9° 00'$ } as in last example.

(3) $b = \left(\begin{matrix} +i.\ \text{sin. } (C \pm T) & + & a.\ \text{vers. } Ia \\ 15.64 & & 54.40 \end{matrix}\right) \div \begin{matrix} \text{vers. } Ib \\ \text{vers. } 17° \end{matrix} = 1603.2.$

(4) $t = \left(\begin{matrix} i.\ \text{cos. } (C \pm T) & - & a.\ \text{sin. } Ia & - & b.\ \text{sin. } Ib \\ 98.77 & & 293.26 & & 468.04 \end{matrix}\right) = 273.6.$

Example 6 (figure 115). *Compound Curves.*—Centres on dif-

ferent sides of **AB**, and $I = 180° - (A' + B')$. From Ta, on a tangent-line **TaTb**, a 5° 30′ curve has been run to a point A with a tangential direction AI. It is required to shift the tangent-point to Tb so that the curve through that point shall pass through B with a tangential direction IB.

i=(**AB**)=200 is the distance from A to B; $A' = 87° 00'$; $B' = 86°$; $I = 180° - (A' + B') = 7° 00'$ is the required change of tangential direction; Ia=44° 00′ is the tangential deflection from Ta to A; a=1042.1 is the radius of the curve. Required the tangential deflection, Ib, from Tb to B; the distance, t, from Ta to Tb, and the radius of the curve through B.

(1) $Ib = Ia - I = 44° - 7° = 37°$.

(2) $(C \pm T) = 180° - (A' + Ia) = B' - Ib = 49° 00'$.

(3) $b = \left(\begin{matrix} -i.\ \sin.\ (C \pm T) & +a.\ \text{vers. } Ia \\ 151.0 & 292.5 \end{matrix}\right) + \left\{\begin{matrix} \text{vers. } Ib \\ \text{vers. } 37° \end{matrix}\right\} = 703.0.$

(4) $t = \left(\begin{matrix} i.\ \cos.\ (C \pm T) & +a.\ \sin.\ Ia & -b.\ \sin.\ Ib \\ 131.2 & 723.9 & 423.1 \end{matrix}\right) = 432.1.$

In the application of these formulæ care must be taken to note the characteristics of the curves, observing whether they are compound or reversed, whether the centres are on the same or different sides of **AB**, whether the curves are convex or concave to **AB**, and whether the triangle AIB, formed by the tangents through A and B and **AB**, is similar to figure 7 or to 7a, so that the change, I, of tangential deflection may be determined in terms of A′ and B′ by the rules of problem V.

Example 7 (figure 117). In this case a curve, with radius a, has been run from a tangent to a point A, and it is required to change the origin of the curve on the tangent so that it shall pass through the *same point*, A, but with a given change, $I = Ib - Ia$, of tangential direction.

$i = 0$; $Ia = 25° 00'$; $a = 2292$; $I = 4° 00'$.

(1) $\left\{\begin{matrix} Ib = Ia + I = 29° 00'. \\ Ia = Ib - I = 25° 00'. \end{matrix}\right.$

(3) $b = a.\ \text{vers. } Ia \div \text{vers. } Ib = 1714.$

(4) $t = a.\ \sin.\ Ia - b.\ \sin.\ Ib = 137.7.$

In these formulæ the longer radius, a, is the radius of the *inside* curve.

Example 8 (figure 118). In this case, the curves terminate, with parallel tangents; $i = (\mathbf{AB})$, is the perpendicular distance between the tangents; $A' = B' = 90°$; $Ia = Ib$, and $(C \pm T) = A - Ia = 90° - Ia$.

$i = 30$; $Ib = Ia = 40°$; $a = 1146$.

(1) $Ia = Ib = 40°$.

(2) $(C \pm T) = 90° - Ia = 50°\ 00'$.

(3) $(a - b) = i$. cos. $Ia \div$ vers. $Ia = i \div$ ex. sec. $Ia = 98.2$.
$b = a - 98.2 = 1047.8$.

(4) $t = (a + i - b)$. sin. $Ib = 82.4 = (a - b)$. tan. Ib.

Note that the longer radius, a, in (3), is the radius of the *inside* curve.

Example 9 (figure 119). In this case, $a = b$, $Ia = Ib$, and it is required to shift the tangent-point so that the terminal tangent may be thrown to the right or left by a perpendicular distance, p.

$$p = 50;\ Ia = Ib = 60°\ 00';$$
$$t = p \div \text{sin. } Ib = 57.7.$$

This value of t is independent of the value of the equal radii. It would, for instance, be the same for 6 stations of a ten-degree curve as for 12 stations of a five-degree curve.

In example 8, t varies with the difference, $(a-b)$, of the two radii.

CHAPTER III.

PROPOSITIONS, DEMONSTRATIONS, AND FORMULÆ.

PROPOSITION I.

In any algebraic sum, or difference of positive and negative quantities, for any one quantity there may be substituted a similar quantity of equal magnitude but contrary sign, PROVIDED *that the sign prefixed be changed from plus to minus, or from minus to plus.*

If, for instance, the deflection SN.AB is to the right and positive, and the equal deflection AB.SN to the left and negative,

$$\pm SN.AB = \mp AB.SN;$$

and if, as stated in proposition V.,

$$AB.CD = +SN.CD - SN.AB,$$

then,
$$AB.CD = +SN.CD + AB.SN = -CD.SN + AB.SN.$$

Such equations are called equivalent, because, although the signs of the terms in the second members may differ, their algebraic sum or difference is the same.

PROPOSITION II.

(*For examples see problem II.*)

The deflection AB.AB, *from any direction to the same direction again, will be either* $\pm$ 360° *or a whole multiple of* $\pm$ 360°; *and, conversely, if the deflection* AB.CD *is equal to*

$\pm$ 360°, *or to any whole multiple of* 360°, AB *and* CD *are one and the same direction.*

It does not, however, follow that the lines on which the points are situated would coincide, because the directions AB and CD may be the same, while the lines **AB** and **CD**, although parallel, may be different.

PROPOSITION III.

(*See problem III., example* 1.)

If the deflection BA.BC *is equal to* $\pm$ 180°, A, B, *and* C *are on the same straight line, and the directions* AB, AC, *and* BC *are the same.*

PROPOSITION IV.

(*For examples see problem III.*)

The deflection AB.YZ, *from an initial direction,* AB, *to a terminal direction,* YZ, *is equal to the algebraic sum of the positive and negative deflections,* AB.CD, CD.EF, . . . , UV.WX, *and* WX.YZ, *made in turning from* AB *to* YZ, *whatever may be the relative positions of the points, and of the intersections of the lines* **AB, CD,** *etc.;* PROVIDED *that the terminal direction,* CD, *of any partial deflection,* AB.CD, *is always the initial direction of the deflection,* CD.EF, *next succeeding.*

This and proposition XIII. are fundamental propositions from which all other propositions of plane linear geometry and trigonometry may be deduced.

This proposition may be expressed by the following formula:

$$AB.YZ = AB.CD + CD.EF + EF.GH + \ldots + UV.WX + WX.YZ. \quad (1)$$

In this formula, deflections in one direction, as to the right, are to be considered positive, and the sign prefixed to any term which may be negative must be changed from plus to minus.

PROPOSITION V.

(*For examples see problem III.*)

The deflection, AB.CD, *from* AB *to* CD, *is equal to the algebraic difference of the bearings to* CD *and* AB *from any third direction* SN, PROVIDED *that the bearing of the initial direction* AB *is made the subtractive term of the difference,* SN.CD − SN.AB.

Formula. AB.CD=SN.CD−SN.AB. (1)

Demonstration. By prop. IV., SN.CD=SN.AB+AB.CD.
Therefore, AB.CD=SN.CD−SN.AB.

PROPOSITION VI.

(*For examples see problem IV.*)

If the bearings to two directions, AB *and* CD, *be measured from different axes,* AN *and* CN′, *the deflection* AB.CD *will be equal to the algebraic difference of the bearings to* CD *and* AB *from their respective axes, plus the deflection from the axis* AN *to the axis* CN′.

Formula. AB.CD=CN′. CD−AN.AB+AN.CN′. (1)

Demonstration. By prop. V., AB.CD =CN′.CD−CN′.AB.
Also, CN′.AB=AN.AB−AN.CN′.

Substituting this value of CN′. AB in the first formula,

AB.CD=CN′. CD−AN.AB+AN.CN′.

If AN is the same direction with CN′, the deflection AN.CN′ is equal to zero, and the formula reduces to the formula, AB.CD=SN.CD−SN.AB, of proposition V.

PROPOSITION VII.

(*For examples see problems II. and III.*)

If the algebraic difference of the bearings to two directions, AB *and* CD, *from any third direction,* SN, *is equal to zero, or to any whole multiple of* 360°, AB *and* CD *are one and the same direction.*

Demonstration. Let n be any whole number, or zero, and $\pm n.360°$ any whole multiple of 360°:

By supposition, $SN.CD-SN.AB=\pm n.360°$.
By proposition V., $SN.CD-SN.AB=AB.CD$.

The deflection from AB to CD is therefore equal to $\pm n.360°$, which is a whole multiple of 360°, and therefore, by proposition II., AB and CD are one and the same direction.

PROPOSITION VIII.

(*For examples see problem II.*)

Two deflections are equivalent when, if measured from the same axis of direction, they differ, algebraically, by any whole multiple of ± 360°, *and of two such equivalent deflections one may be substituted for the other in any trigonometric formula.*

Because the substitution, in the second member of an equation, of one equivalent deflection for another, will result in an equivalent expression for the quantity computed in the first member.

It will be found convenient, when the measured or calculated value of an angle is greater than 180°, to substitute an equivalent value less than 180°, determined by the rules of problem II., to which reference is made for examples of the application of this proposition.

As the difference between two angles of 180° will be either o or 360°, the sign of 180°, when a term of any trigonometric quation, may be always changed from plus to minus or from nus to plus.

As every deflection may be expressed by values which, although equivalent, may differ in sign and magnitude, the symbol of equality between two deflections is to be understood in the sense of equivalence. By the equation, AB.CD=EF.GH, for instance, it is to be understood that the inclination between the lines **AB** and **CD** is the same as between the lines **EF** and **GH**, but the same cannot be alleged with regard to the deflections, because they may differ by any whole multiple of 360°.

The equality of the inclinations is not, however, to be understood in the mathematical sense, for inclinations are not quantities, and one inclination cannot be a part or multiple of another.

PROPOSITION IX.

The deflection from AB *to* CD *is equal to the deflection from the direction* BA, *opposite to* AB, *to the direction* DC, *opposite to* CD.

Formula. AB.CD=BA.DC. (1)

Demonstration. By prop. IV., AB.CD=AB.BA+BA.DC+DC.CD=±180°+BA.DC±180°= BA.DC±(0 or ±360°), and therefore, by proposition VIII., AB.CD is equal, or equivalent, to BA.CD.

In plane geometry, every line is supposed to make four angles with every intersecting line, and the angles AB.CD and BA.DC are called the alternate angles.

PROPOSITION X.

TANGENTIAL AND RADIAL DEFLECTIONS.

(*For examples see problem V.*)

(Problem V., figures 7 and 7a.) Two tangent lines through A and B intersect at I; A′ and B′ are the acute angles between **AB** and the tangent lines; the turn from one tangential direction to the other will be (A′± B′), or 180°− (A′± B′), =I.

(Plates 17 to 22.) This turn is to be made over two circular arcs, ATa and TbB, having **TaTb** as a common tangent, Ia representing the tangential deflection over the arc ATa, and Ib over the arc TbB.

The curves are reversed when, as in figure 90, the centres, C and Z, are on different sides of the *common tangent*, and compound when, as in figures 91 and 92, the centres are on the same side.

The centres are on different sides of **AB** in figures 104 and 105, and on the same side in figures 106 and 107.

$$\left.\begin{array}{l} \mathrm{I}=\mathrm{I}a+\mathrm{I}b=(\mathrm{A}'\pm\mathrm{B}') \\ \text{When the curves are reversed, with centres on different sides of } \mathbf{AB}\text{; or when the curves are compound, with centres on the same side of } \mathbf{AB}. \\ \mathrm{I}=\mathrm{I}a+\mathrm{I}b=180^\circ-(\mathrm{A}'\pm\mathrm{B}') \\ \text{When the curves are compound, with centres on different sides of } \mathbf{AB}\text{; or when the curves are reversed, with centres on the same side of } \mathbf{AB}. \end{array}\right\} \quad (1)$$

Demonstration. In any single curve of which C is the centre, CA the initial, and CTa the terminal radius, and Ia=AI.ITa the *tangential* deflection, Ia will be equal to the deflection *from* the initial *to* the terminal radial direction, PROVIDED that both are estimated *from* centre *to* tangent, or *from* tangent *to* centre.

This relation may be expressed by the following formula:

$$\mathrm{I}a=\mathrm{AI}.\mathrm{IT}a=\mathrm{IA}.\mathrm{T}a\mathrm{I}=\mathrm{CA}.\mathrm{CT}a=\mathrm{T}a\mathrm{C}.\mathrm{AC}. \quad (2)$$

In any of the figures, as 104 and 112, in which the two arcs, ATa and TbB, are connected by the common tangent **TaTb**, let the radius through the initial point A be estimated in the direction CA, *from* centre *to* tangent, and the radius through the terminal point B, in the direction BZ, *from* tangent *to* centre (Z being the centre of the curve TbB), and let R= CA.BZ be the deflection from the initial to the terminal radial direction thus estimated.

Prop. IV. (1). $\mathrm{R}=\mathrm{CA}.\mathrm{BZ}=\mathrm{CA}.\mathrm{CT}a+\mathrm{CT}a.\mathrm{T}b\mathrm{Z}+\mathrm{T}b\mathrm{Z}.\mathrm{BZ}$.

Prop. X. (1). $\mathrm{CA}.\mathrm{CT}a=\mathrm{I}a$, and $\mathrm{T}b\mathrm{B}.\mathrm{BZ}=\mathrm{I}b$, and as $\mathrm{I}a+\mathrm{I}b=\mathrm{I}$, $\mathrm{R}=\mathrm{I}+\mathrm{CT}a.\mathrm{T}b\mathrm{Z}$. (3)

The radii, CT_a and T$_b$Z, being perpendicular to the common tangent **T$_a$T$_b$**, are parallel. When the curves are reversed, as in figure 104, the directions CT_a and T$_b$Z are the same, and CT_a.T$_b$Z=0. When the curves are compound, as in figure 112, CT_a and T$_b$Z are opposite directions, and, therefore, CT_a.T$_b$Z=180°.

Substituting these values of CT_a.T$_b$Z, in (3),

$$I = R \text{ for reversed curves, and } 180° + R \text{ for compound curves.} \quad (4)$$

If CN, figures 93 and 94, be a direction from C *towards*, and *at right angles to*, **AB**, then, as **CN** and **CA** are perpendicular to **AB** and **AI**, and **CN** and **BZ** to **AB** and **BI**,

$$CN.CA = AB.AI = \pm A'. \quad \text{(Figures 93 and 94.)}$$

$$CN.BZ = AB.BI = \begin{cases} \pm B' \text{ for centres on different sides of } \mathbf{AB}. \text{ (Figure 94.)} \\ 180° \pm B' \text{ for centres on the same side of } \mathbf{AB}. \text{ (Figure 93.)} \end{cases}$$

Prop. V. (1). CN.BZ−CN.CA=CA.BZ=R, and substituting for CN.BZ−CN.CA its value from the above equations,

$$R = \begin{cases} (A' \pm B') \text{ for centres on different sides of } \mathbf{AB}, = I \text{ for reversed curves.} \\ 180° - (A' \pm B') \text{ for centres on the same side of } \mathbf{AB}, \\ \quad = 180° - I \text{ for compound curves.} \end{cases}$$

From this equation the values of I in (1) are deduced.

PROPOSITION XI.

A line is straight when the directions from one point to others on the line are either the same or opposite one to the other.

Let A, B, C, . . . , X, Y, and Z, be points on the same straight line; (**AB**), (**BC**), (**CD**), . . . , (**XY**), and (**YZ**), distances from A to Z, measured *to* and *from* each of the points B, C, D, . . . , X, and Y, and let these distances be

positive when measured in one direction, as AB, and negative when measured in the opposite direction, BA.

The distance (**AZ**) *will be equal to the algebraic sum of the distances* (**AB**), (**BC**), (**CD**), . . . , (**XY**), *and* (**YZ**).

The distance (**BY**) FROM *a point* B TO *a point* Y *will be equal to the algebraic difference of the distances* TO Y *and* B FROM *any third point* O.

These propositions may be expressed by the following equations:

$$(\mathbf{AZ})=(\mathbf{AB})+(\mathbf{BC})+(\mathbf{CD})+\ \ldots\ +(\mathbf{XY})+(\mathbf{YZ}). \quad (1)$$

$$(\mathbf{BY})=(\mathbf{OY})-(\mathbf{OB}). \quad (2)$$

In each of these equations, the sign prefixed to any negative term must be changed from plus to minus, or from minus to plus, and the term in the first member of the equation will then be equal to the difference between the sums of the plus and minus terms in the second member, with the sign of the greater sum.

In (2), when B is the initial point of the distance (**BY**) in the first member, (**OB**) must be the subtractive term in the second member.

PROPOSITION XII.

(*Plate* 2, *figure* 11.)

Two straight lines, **AB** and **CD**, are parallel when the directions, AB and CD, are either the same or opposite one to the other.

Two parallel lines will not meet, how far soever they may be produced, and all distances, MEASURED IN THE SAME DIRECTION, *between the lines, will be equal one to the other.*

If, therefore, **ANa**, **BNb**, **CNc**, . . . , **YNy**, *and* **ZNz** *are a series of parallel lines, the distance from* **ANa** *to* **ZNz**, *measured in any direction, will be equal to the algebraic sum of the distances measured in the same direction, from* **ANa** *to* **BNb**, *from* **BNb** *to* **CNc**, *from* **CNc** *to* **DNd**, . . . , *and* **YNy** *to* **ZNz**.

Demonstration.—Assuming the equidistance of parallel , this proposition may be deduced from proposition XI.;

for if the straight line, **AZ**, be drawn across the parallel lines, cutting them at A, B, C, etc., the distances from one parallel line to another, measured on this line, will be equal to the distances (**AB**), (**BC**), (**CD**), etc., and, by proposition XI., the algebraic sum of these distances is equal to the distance, (**AZ**), from the line **AN***a* to the line **ZN***z*.

In estimating these distances, those measured in one direction, as AB, are positive, and those measured in the opposite direction, BA, are negative, and the sign of the algebraic sum will show whether the line **ZN***z* is right or left of **AN***a*.

PROPOSITION XIII.

DEPARTURES, LATITUDES, SINES, COSINES, ETC.

(Demonstration of problems X., XI., and XII.)

1. DEFINITIONS.—*Positive and Negative Directions.*—In this proposition, the right will be positive and the left negative, with reference to deflections or other quantities.

Rectangular Co-ordinate Axes.—These axes are represented by the symbols **SN** and **EW**, and their intersection is called the *origin of co-ordinates.* The directions SN and EW have the same relation as north and west, the deflection from SN to EW being 90 degrees to the right, or $+90°$, and, therefore, these terms will be used to indicate the directions of the co-ordinate axes; east being right, and west left, of **SN**, and north right, and south left, of **EW**.

The letters N and W affixed to a letter representing a point will indicate a direction, or line, through the point, parallel to the axes **SN** and **EW**. Thus, for instance, **AN***a* and **BN***b* are lines through A and B, parallel to **SN**, and **AW***a* and **BW***b* lines parallel to **EW**.

The co-ordinates of a point are the perpendicular distances, right or left, *from* the axes *to* the point, positive when measured to the right, and negative when measured to the left.

The distance from **SN** is the *ordinate*, and the distance from **EW** *the abscissa.*

e_B is the ordinate, and n_B the abscissa, of a point B, one being

the distance east or west of **SN**, and the other the distance north or south of **EW**.

Departures and Latitudes (figure 19).—A is the initial and B the terminal point of a line (**AB**); **AN***a* and **BN***b* are lines through A and B, parallel to **SN**, and **AW***a* and **BW***b* lines parallel to **EW**.

The departure, d_a, of (**AB**) is the perpendicular distance from **AN***a* to **BN***b*, and the latitude, l_a, the perpendicular distance from **AW***a* to **BW***b*. The departure is, therefore, the distance by which the terminal point is east or west of the initial point, and the latitude the distance north or south. As, therefore, by whatever distance A is to the right of B, by that same distance B is left of A, it follows that the departure and latitude of (**AB**) are equal in magnitude, but contrary in sign, to the departure and latitude of (**BA**), of which B is the initial point.

The line is, therefore, the hypothenuse of a right-angled triangle of which the departure and latitude are the perpendiculars.

2. *Algebraic Sums and Differences of Departures and Latitudes* (figure 19).—(**AB**) and (**BC**) are two lines, and (**AC**) is the *closing* line FROM A TO C. Through A, as the origin of co-ordinates, and through B and C, lines are drawn parallel to the co-ordinate axes. The departures of (**AB**), (**BC**), and (**AC**) are the perpendicular distances from **AN***a* to **BN***b*, from **BN***b* to **CN***c*, and from **AN***a* to **CN***c*, and the latitudes are the perpendicular distances from **AW***a* to **BW***b* and from **BW***b* to **CW***c*. If, therefore, d_a and d_b are the departures, and l_a and l_b the latitudes, of (**AB**) and (**BC**), and e_c and n_c the co-ordinates of the point C, equal to the departure and latitude of the line (**AC**),

Problem XI. (1). $e_c = d_a + d_b$. $n_c = l_a + l_b$. (a)
(2). $d_b = e_c - d_a$. $l_b = n_c - l_a$. (b)

As (a) will hold good, whatever may be the number of lines, (**AB**), (**BC**), . . . , (**XY**), and (**YZ**), between the initial and terminal points of the series, *the departure of the closing line*, (**AZ**), *is equal to the algebraic sum of the departures, and the latitude to the algebraic sum of the latitudes, of the several lines of the series. From* (b) *it follows that the departure of a line,*

(**BC**), *is equal to the algebraic difference of the departures, and the latitude to the algebraic difference of the latitudes, of any two lines,* (**AB**) *and* (**AC**), *from any third point,* A, *to the extremities of* (**BC**), PROVIDED *that the departure and latitude of the line,* (**AB**), *through the initial point of* (**BC**), *are the subtractive terms of the algebraic differences.* These propositions may be illustrated by supposing a man to move from a point A to a point Z, over a path consisting of a series of broken lines or curves. The total departure would be the difference of the eastings and westings of the several lines, and the total latitude the difference of the northings and southings. It remains to consider in what manner the departures and latitudes of the several lines may be represented as quantities which will be parts of the departure and latitude of the closing line.

3. *Sines, Cosines, Tangents, and Cotangents* (figure 18).—a is the length, d the departure, and l the latitude of a line (**AB**), and E and N are the bearings from the directions, AN and AW, of the co-ordinate axes. *These bearings are to be estimated, right or left, from* 0° *to* 180°, *and not by quadrant, or so as to exceed* 180°. Divide the line (**AB**) into "a" equal parts, each of length unity, and through the extremities of these parts draw lines parallel to the co-ordinate axes. The distance, in the direction AB, from each line to the parallel line next succeeding, is unity, and the distances, at right angles to the co-ordinate axes, are therefore equal to each other. The distances at right angles to the axis **AN** are the departures of the line (**AB**) in the distance unity, or *the rates of divergence of the line* (**AB**) *from the axis* **AN**, and this rate of divergence is called *the sine of the bearing* E, or *sin.* E, and the total departure in the distance a is equal to "a" times the departure in the distance unity, or to $a.sin.$ E.

In the same manner, the divergence of (**AB**) from the axis **AW**, or the latitude, is equal to "a" times the sine of N, or to $a.sin.$ N.

It is evident that B will be to the right of the axis through A when the bearing of AB is to the right of that axis, and to the left when the bearing is to the left; and, therefore, *the sign of the departure will be the same as the sign of the bearing,* E, *and the sign of the latitude as the sign of* N.

By definition, AW.AN=(+90°); and by proposition IV.,

$$AW.AB = N = AW.AN + AN.AB = (+90°) + (\pm E).$$

It follows, therefore, that N and the latitude are *positive* for all values of E *less* than 90°, and *negative* for values *greater* than 90°, whether E is positive or negative.

It is, in fact, sufficiently evident, without demonstration, that B will lie to the right, or north, of **AW**, when the bearing of AB from AN, whether to the right or left, is less than 90°, and to the left, or south, when that bearing is greater than 90°.

The bearing, N, from AW is the *complement* of the bearing, E, from the axis AN, and the *sine of* N is called the *cosine of* E, and, therefore, *estimating the bearings from* AN, the rule for the signs of sines and departures, and cosines and latitudes, will be as follows: *Sines and departures have the same signs with the bearings, and cosines and latitudes are positive when the bearings are less than* 90°, *and negative when the bearings are greater than* 90°.

Tangents and cotangents.—The tangent of E is the ratio of the departure to the latitude, or of the sine to the cosine, and the cotangent is the tangent of the complement, $90° \pm E$, and, therefore,

$$\text{Tan. } E = (a.\sin. E) \div (a.\cos. E) = (\sin. E \div \cos. E).$$

$$\text{Cot. } E = \sin.(90° \pm E) \div \cos.(90° \pm E) = (\cos. E \div \sin. E) = \frac{1}{\text{tan. } E}$$

Sines and departures are measured at right angles to an axis the direction of which does not change with the directions of the lines, and, therefore, the departure of the closing line may be regarded as composed of the departures of the several lines between its extremities.

A tangent may be regarded as the rate of divergence, measured at right angles with the divergent line (figure 20).

As, therefore, the directions of the tangential departures change with the directions of the divergent lines, the tangential departure of the closing line is not equal to the sum of the tangential departures of the several lines between its extremities.

The signs of tangents are, therefore, to be determined from the signs of the sine and cosines, on the principle that the sign of the quotient is *positive* when the signs of the nume-

rator and denominator are *alike*, and *negative* when they are *unlike*.

The signs of tangents and cotangents are therefore to be determined by the following rules:

A tangent or cotangent is positive when the signs of the sine and cosine are alike, and negative when they are unlike, and, therefore, tangents and cotangents have like signs to bearings less than 90°, *and unlike signs to bearings greater than* 90°.

In all trigonometric formulæ, the sums and differences are algebraic, and, in applying the formulæ, the sign prefixed to a term which is negative must be changed.

When the term is a single quantity, the signs are determined by the above rules; but if the term is a multiple or quotient of two factors, it will be positive when the signs of the factors are alike, and negative when they are unlike.

Sines, cosines, etc., are generally treated as functions of circular arcs, and, as they may be regarded as rates of divergence, or divergencies in a distance unity, they may be readily represented by means of a circle with a radius unity, but they are, in reality, ratios between the lengths of straight lines, and, like angles, have no necessary connection with circles.

4. *Formulæ of problem XI.*—These formulæ are based upon the following propositions, which have been explained in the preceding articles:

The closing line, from the initial to the terminal point of a series of intersecting lines, is equal to the hypothenuse of a right-angled triangle, the perpendicular of which is equal to the algebraic sum of the departures, and the base to the algebraic sum of the latitudes, of the several lines composing the series. The bearing of the closing line is equal to the acute angle of this triangle, opposite to the perpendicular, when the sum of the latitudes is positive, and to the supplement of that angle when the sum of the latitudes is negative.

The sign of the bearing is the same with the sign of the sum of the departures.

Formula (3) is based on the following proposition deduced from the above:

The departure of a line, (**FC**), *is equal to the algebraic difference of the departures, and the latitude to the algebraic difference of the latitudes, of two lines,* (**AF**) *and* (**AC**), *from any third point,* **A**, *to the extremities of* (**FC**), PROVIDED *that the*

departure and latitude of the line (**AF**), *through the initial point*, F, *of* (**BC**), *be made the subtractive terms of the algebraic differences.*

These propositions may be expressed by the following formulæ:

(**AB**), (**BC**), (**CD**), . . . , (**XY**), and (**YZ**) are a series of intersecting lines; a, b, c, . . . , x, and y, are the lengths, and A, B, C, . . . , X, and Y the bearings, of the several lines, of which A, B, C, . . . , X, and Y are the initial points. Z is the bearing, and z the length, of the closing line from A to Z.

b. sin. B is the departure, and b. cos. B the latitude, of the line (**BC**), and $e_z=(z.\sin.\ Z)$ is the departure, and $n_z=(z.\cos.\ Z)$ is the latitude, of the closing line (**AZ**).

The departure and latitude, and length and bearing, of the closing line are determined by the following formulæ:

$$\left.\begin{aligned} e_z &= a.\sin.\ A + b.\sin.\ B + \ldots + x.\sin.\ X + y.\sin.\ Y \\ &= \Sigma d_{y.}^{a.} \\ n_z &= a.\cos.\ A + b.\cos.\ B + \ldots + x.\cos.\ X + y.\cos.\ Y \\ &= \Sigma l_{y.}^{a.} \end{aligned}\right\} \quad (1)$$

From the departure and latitude thus determined, the bearing and length are obtained by either of the formulæ (2) or (3). The latter is based upon the proposition, demonstrated in article 5, that the square of the hypothenuse is equal to the sum of the squares of the perpendicular sides of a right-angled triangle.

$$\left.\begin{aligned} &\text{Tan. } Z = e_z \div n_z. \\ &z = e_z \div \sin.\ Z = n_z \div \cos.\ Z. \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} &z = (e_z^2 + n_z^2)^{\frac{1}{2}}. \\ &\sin.\ Z = (e_z \div z). \quad \cos.\ Z = (n_z \div z). \end{aligned}\right\} \quad (3)$$

(**AF**) and (**AC**) are two lines extending from A to the extremities of a third line, (**FC**), of which F is the initial point.

e_F and n_F is the departure and latitude of (**AF**), e_C and n_C of (**AC**), and d_f and l_f of (**FC**). The length and bearing of (**FC**) may be determined by (5) or by (6).

$$d_f = (e_C - e_F). \qquad l_f = (n_C - n_F). \quad (4)$$

$$\left.\begin{aligned} &\text{Tan. SN.FC} = (e_C - e_F) \div (n_C - n_F). \\ &(\mathbf{FC}) = (e_C - e_F) \div \sin.\ \text{SN.FC} = (n_C - n_F) \div \cos.\ \text{SN.FC}. \end{aligned}\right\} \quad (5)$$

$$\left.\begin{array}{l}(\mathbf{FC}) = \sqrt{(e_C - e_F)^2 + (n_C - n_F)^2}. \\ \text{sin. SN.FC} = (e_C \div e) \div (\mathbf{FC}). \\ \text{cos. SN.FC} = (n_C - n_F) \div (\mathbf{FC}).\end{array}\right\} \quad (6)$$

In (1), departures have the same signs with the bearings, and latitudes are negative when the bearings are greater than 90°.

In (2) and (3), if Z′ is the acute angle corresponding to the numerical value of the tangent, Z is equal to Z′ when the latitude, n_z, is positive, and to (180°−Z′) when n_z is negative.

In (4), the departure, e_F, and the latitude, n_F, of the line (**AF**), through the initial point of (**FC**), are the subtractive terms of the algebraic differences in the second members of the equations. In (5) and (6), Z is obtuse when $(n_C - n_F)$ is negative, and has the same sign with $(e_C - e_F)$.

5. *Problem XII. Triangles* have three sides and three angles, and if a side and two other parts are given the remaining parts may be determined.

In the solution of the general problem there will be four different cases, in which the parts given are as follows:

(1) A side and two angles.
(2) Two sides and an opposite angle.
(3) Two sides and an included angle.
(4) Three sides.

(Figures 25 and 26.) A, B, and C are the three angles, A being obtuse in figure 26.

The three sides are, (**CB**)=a, opposite A; (**AC**)=b, opposite B; and (**AB**)=c, opposite C.

If two angles are given, the third may be determined by the formula,

$$A+B+C=180°. \quad (a)$$

(Figure 22.) Refer the sides to any pair of rectangular axes, and let (**AB**) be the *closing line* to the sides (**AC**) and (**CB**).

(1) Article 4. Dep. of (**AB**)=dep. of (**AC**)+dep. of (**CB**).
Lat. of (**AB**)= lat. of (**AC**) + lat. of (**CB**).

Assuming A as the origin, and **AB** as the axis, of co-ordinates, the bearings, lengths, departures, and latitudes of the lines are as follows:

AB.AC=A.	(**AC**)=b.	dep.=b.sin. A.	lat.=b.cos.A.
AB.CB=(−B).	(**CB**)=a.	dep.=(−a.sin.B)	lat.=a.cos.B.
AB.AB=0.	(**AB**)=c.	dep.=0.	lat.=c.cos.0=c.

In these equations, the departures, a. cos. B and b. cos. A, will be negative when the angles are greater than 90°.

Equating the departure and latitude of the closing line, (**AB**), with the algebraic sum of the departures and latitudes of (**AC**) and (**CB**), the following equations are deduced:

(2) Article 4. b. sin. A−a.sin. B=0. b. cos. A+a.cos. B=c.

Therefore, a.sin. B=b.sin. A. (b)
a.cos. B=c−b.cos. A. (c)
Tan. B=b.sin. A÷(c−b.cos. A). (d)

Cases 1 *and* 2. When a side and two angles, or two sides and an opposite angle, are given, the remaining parts may be determined by (a) and (b), as shown in (9) and (10), problem XIII.

Case 3. When two sides, a and b, are given, and the included angle C, the angle B is determined by (d), changing the sign of b.cos. B in the denominator when A is greater than 90°.

Formula (11), problem XIII., is, however, better suited for calculation by logarithms, and is obtained in the following manner:

(Figure 27.) a and b are the given sides, and the axis **CN**, from C as the origin, bisects the included angle C.

By (4), article 4, the departure and latitude of (**AB**) are equal to the algebraic differences of the departures and latitudes of two lines, (**CA**) and (**CB**), from any point C to the extremities of (**AB**).

Calling right negative and left positive, and representing the length and bearing of (**AB**) by Z and z, the bearings, lengths, departures, and latitudes of the sides of the triangle are as follows:

CN.AB = Z. (**AB**) = z. dep. = z. sin. Z. lat. = z. cos. Z.
CN.CA = (−½ C). (**CA**) = b. dep. = (−b. sin. ½ C). lat. = (+b. cos. ½ C).
CN.CB = (+½ C). (**CB**)=a. dep. =(+a.sin. ½ C). lat. = (+a. cos. ½ C).

Equating the differences of the departures and latitudes of (**CB**) and (**CA**), with the departures and latitude of (**AB**):

Art. 4. (4) $z.\sin. Z=(a+b).\sin. \frac{1}{2}C.$ $z.\cos. Z=(a-b).\cos. \frac{1}{2}C.$
(5) Tan. $Z=\tan. \frac{1}{2}C.(a+b)\div(a-b).$
(6) $z=(a+b).\sin. \frac{1}{2}C\div\sin. Z=(a-b).\cos. \frac{1}{2}C\div\cos. Z.$

The side, $z=c$, being thus determined, A and B may be calculated by (b), or directly from Z, in the following manner:

As cot. $Z=1\div\tan. Z$, and cot. $\frac{1}{2}C=1\div\tan. \frac{1}{2}C$, the equation above may be placed under the following form:

Cot. $Z=\cot. \frac{1}{2}C.(a-b)\div(a+b).$ (e)
$C=180°-(A+B)$, and, therefore, $\frac{1}{2}C=90°-\frac{1}{2}(A+B).$
$Z=CN.AB=CN.CB+CB.AB=\frac{1}{2}C+B=90°-\frac{1}{2}(A-B).$

Therefore cot. $Z=\tan. \frac{1}{2}(A-B)$, and cot. $\frac{1}{2}C=\tan. \frac{1}{2}(A-B)$, and substituting these values in (e),

Tan. $\frac{1}{2}(A-B)=\tan. \frac{1}{2}(A+B).$ $(a-b)\div(a+b).$ (f)
$A=\frac{1}{2}(A+B)+\frac{1}{2}(A-B).$ $B=\frac{1}{2}(A+B)-\frac{1}{2}(A-B).$

Case 4. For the solution in this case, it is necessary to prove that *the square of the length of the hypothenuse of a right-angled triangle is equal to the sum of the squares of the lengths of the base and perpendicular.*

This is generally deduced from the proposition that *the area of the square described on the hypothenuse is equal to the sum of the areas of the squares described on the other sides.*

The squares *described on the sides* are not, however, the squares of the lengths of the sides, and, although one proposition may be deduced from the other, they are not identical, and the proposition may be demonstrated without reference to units of area.

(Figure 22) (c). $c=a.\cos. B+b.\cos. A$, and multiplying both members of the equation by c,

$$c^2=c.\ a.\ \cos. B+c.\ b.\ \cos. A. \quad (g)$$

When C is a right angle, $A=90°-B$, and cos. $B=\sin. A$, and substituting this value of cos. B in (g),

$$c^2=a.\ c.\ \sin. A+b.\ c.\ \cos. A. \quad (h)$$

In this case, c is the hypothenuse; $a=c.\sin.\ A$; $b=c.\cos.\ A$, and substituting these values in (h),

$$c^2=a^2+b^2. \quad \text{(i)}$$

By means of this proposition, the formula for case 4 may be obtained as follows:

(Figure 22.) From the vertex, C, of the triangle ACB, let fall on the base, (**AB**), the perpendicular, (**CW**)$=p$, dividing the base into two segments, (**WB**)$=y$ and (**WA**)$=z$.

In the two right-angled triangles, ACW and BCW,

$$b^2=z^2+p^2. \qquad a^2=y^2+p^2.$$

Subtracting one equation from the other, $(b^2-a^2)=(z^2-y^2)$, and, therefore,

$$(z-y)=(b+a)\,.\,(b-a)\div(z+y)=c, \text{ and } \tfrac{1}{2}\,(z+y)+\tfrac{1}{2}\,(z-y)=z,$$
$$\text{and } \tfrac{1}{2}\,(z+y)-\tfrac{1}{2}\,(z-y)=y.$$

z is the base of the right-angled triangle CAW, and y the base of the right-angled triangle CBW, and, therefore,

$$\text{Cos. A}=z\div b, \text{ and cos. B}=y\div a.$$

Substituting x for $(z-y)$, this is the same with (12), problem XII.

5. *Trigonometrical Formulæ.*—The following formulæ are identical with, or deduced from, the formulæ of the preceding article:

(b) $a.\sin.\ B=b.\sin.\ A.$ (c) $c=a.\cos.\ B+b.\cos.\ A.$

(f) $(a-b)\div(a+b)=\tan.\ \frac{1}{2}\ (A-B)\div\tan.\ \frac{1}{2}\ (A+B).$

(i) $c^2=a^2+b^2$, if $C=90°$.

If a, in (b), be made equal to sin. A, b will be equal to sin. B, and, in the same manner, it may be shown that c will be equal to sin. C.

Therefore, as one angle is the supplement of the sum of the two others, and the sine of the angle is equal to the sine of the supplement, there may be substituted in the formulæ for the sides of the triangle, sin. A = sin. (B + C) for a; sin. B = sin. (A+C) for b; and sin. C=sin. (A + B) for c, as shown in figure 23.

The following equations are obtained from the above by making these substitutions, with the understanding that the

substitutions are to be made for *every* side which enters into the equation, and not for one side only:

From (c), sin. C=sin. (A+B)=sin. A.cos. B+sin. B.cos. A.
From (f), (sin. A—sin. B) ÷ (sin. A + sin. B) = tan. ½ (A—B) ÷tan. ½ (A+B).

In (f), if B=90°—A, C=90°; cos. B=sin. A; sin. B=cos. A, and substituting these values in (c):

$$\text{sin. } (C=90°)=1=\text{sin.}^2A+\text{cos.}^2A.$$

From this formula may be deduced the proposition that the square of the hypothenuse of a right-angled triangle is equal to the sum of the squares of the other sides, and, conversely, assuming this proposition, the formula may be obtained by substituting, in (i), sin. (C = 90°) for *c*, sin. A for *a*, and sin. B = cos. A for *b*.

These, and the formulæ of preceding articles, are given as examples of the application of the theory of departures and latitudes.

In the succeeding article there are given, for reference, such formulæ as may be required for the demonstrations of the problems of chapter III.

6. *Formulæ for Reference in Demonstrations of Problems.*—In these formulæ, the signs prefixed to the terms are applicable only when the terms are positive, and must be changed when they are negative.

The angles are to be considered as bearings, estimated, right or left, from 0° to 180°, and not by quadrant, or to exceed 180°.

Sines are negative when the bearings are negative.

Cosines are negative when the bearings are greater than 90°.

Tangents and cotangents have like signs to bearings less than 90°, and unlike signs to bearings greater than 90°.

$$\text{sin. } A=2 \text{ sin. } \tfrac{1}{2} A \text{ . cos. } \tfrac{1}{2} A. \quad (7)$$

$$\text{tan. } A = \text{sin. } A \div \text{cos. } A. \quad (8)$$

$$\text{vers. } A=1-\text{cos. } A=2 \text{ sin.}^2 \tfrac{1}{2} A. \quad (9)$$

$$\text{vers. } (180°-A)=1+\text{cos. } A=2 \text{ cos.}^2 \tfrac{1}{2} A. \quad (10)$$

$$\text{sin. } (A \pm B)=\text{sin. } A \text{ cos. } B \pm \text{cos. } A \text{ sin. } B. \quad (11)$$

$$\text{cos. } (A \pm B)=\text{cos. } A \text{ cos. } B \mp \text{sin. } A \text{ sin. } B. \quad (12)$$

$$\text{sin. A} + \text{sin. B} = 2 \text{ sin. } \tfrac{1}{2}(\text{A}+\text{B}). \text{ cos. } \tfrac{1}{2}(\text{A}-\text{B}). \quad (13)$$
$$\text{sin. A} - \text{sin. B} = 2 \text{ sin. } \tfrac{1}{2}(\text{A}-\text{B}). \text{ cos. } \tfrac{1}{2}(\text{A}+\text{B}). \quad (14)$$
$$\text{cos. B} + \text{cos. A} = 2 \text{ cos. } \tfrac{1}{2}(\text{A}+\text{B}). \text{ cos. } \tfrac{1}{2}(\text{A}-\text{B}). \quad (15)$$
$$\text{cos. B} - \text{cos. A} = 2 \text{ sin. } \tfrac{1}{2}(\text{A}+\text{B}). \text{ sin. } \tfrac{1}{2}(\text{A}-\text{B}). \quad (16)$$
$$\text{tan. A} \pm \text{tan. B} = \text{sin. } (\text{A} \pm \text{B}) \div \text{cos. A cos. B}. \quad (17)$$
$$\text{cot. B} \pm \text{cot. A} = \text{sin. } (\text{A} \pm \text{B}) \div \text{sin. A sin. B}. \quad (18)$$

PROPOSITION XIV.

(Demonstration of problem XVI., figures 47 to 63.)

1. All problems in relation to curves intersecting at given angles, or touching at points of compound or reversed curvature, may be solved by means of the following proposition:

If, with any radius, a circle be drawn through two points, A *and* B, *then every pair of curves from* A *and* B, *crossing on that circle, will intersect at the same angle, if from the same pair of tangent-lines at* A *and* B; *but if, from different pairs of tangent-lines, intersecting at* I_1 *and* I_2, *two pairs of curves be drawn to intersection at* F_1 *and* F_2 *on the circle through* A *and* B, *the algebraic sum,* (F_1+I_1), *of the two deflections between the pairs of tangents intersecting at* F_1 *and* I_1 *will be equal to the algebraic sum,* (F_2+I_2), *of the deflections between the pairs of tangents intersecting at* F_2 *and* I_2.

2. *Demonstration* (figure 47). Through **A and B, from the** same or different pairs of tangents, draw to intersection on the circle AFB, at F_1 and F_2, two pairs of curves, and from F_1 and F_2 draw chords to A and B, forming the triangles AF_1B and AF_2B with a common base (**AB**), and let O be the centre of the circle AFB, and K_1 and K_2 the angles, at the vertices of the triangles, subtended by the arc AB.

By (2), problem VIII., the deflection between the chords intersecting at K_1 is equal to $\frac{1}{2}(I_1+F_1)$, and the deflection between the chords intersecting at F_2, to $\frac{1}{2}(I_2+F_2)$; and as the two pairs of chords intersect on the circumference of the circle AFB, and the angles between them are subtended by the same arc, AB, they are each equal to half the central angle, AOB, subtended by that arc, and, therefore, if $r = (\mathbf{OA}) =$

(OB) = **(OF)** is the radius of the circle ABF, and **(AB)** = i is the chord subtending the angles K_1 and K_2,

$$(I_1 + F_1) = (I_2 + F_2) = AOB. \quad (a)$$
$$F_2 = F_1 + (I_1 - I_2). \quad (b)$$
$$r = \tfrac{1}{2} i \div \sin. \tfrac{1}{2} (I + F). \quad (c)$$

From (b) it follows, if I_1 is equal to I_2, that F_1 will be equal to F_2, and, therefore, every pair of curves from the same tangential *directions* will intersect on the circle AFB at the same angle.

If the two pairs of curves are from the same tangent-*lines*, but with centres on the same side of **AB** in one case, and on different sides in the other, as for the two pairs intersecting at F_1 and F_2, figure 47, then, by problem V., $I_2 = -(180° - I_1)$, which, by problem II., is equivalent to $180° + I_1$, and substituting this value of I_2 in (b) gives F_2 equal to $180° - F_1$, so that, although the *deflections* F_1 and F_2 are supplements one of the other, the *angles* are the same.

It therefore follows that all pairs of curves, from the same tangent-lines, whether I_1 is equal to I_2 or to $180° - I_2$, will cross on the circumference of the circle AFB at the same *angle*.

If, in (b), $I_2 = (I_1 + F_1)$ or $180° - (I_1 + F_1)$, then F_2 will be equal to 0° or to 180°, and, therefore, F_2 will be a point of compound or reversed curvature for the pair of curves from the tangents crossing at I_2. In figure 47, for instance, the curves cross at an angle F_1, and I_1 is equal to $(A'_1 - B'_1)$. In figure 48, the direction of the tangent through B is changed, so that the angle B'_2 is equal to $(B'_1 - F_1)$, and, therefore, I_2 is equal to $(A'_1 - B'_1) + F_1$, and F_2 to zero. All pairs of curves, from the tangents through A and B, will, therefore, touch at points of compound or reversed curvature on the circle $A_2F_2B_2$.

It does not, however, follow, because all pairs of curves from the same tangents, and crossing on the circle AFB, will intersect at the same angle, that all curves which intersect at that angle will cross on the same circle, because the sum, (I + F), is algebraic, and every pair of curves intersects at two points, for which the signs of F are different, although the magnitudes are the same, and, therefore, the angle between the chords from A and B to the intersection at F, and the ius. r, of the circle AFB, will be $\tfrac{1}{2}$ (I+F) and $\tfrac{1}{2} i \div \sin. \tfrac{1}{2}$ (I+

in one case, and $\frac{1}{2}$ (I−F) and $\frac{1}{2}\,i \div$ sin. $\frac{1}{2}$ (I−F) in the other. In figure 51, for instance, the *angles* between the crossing tangents of the four pairs of curves are the same in magnitude, but the *signs* of the *deflections* are different on the upper and lower circles, the radii being $\frac{1}{2}\,i \div$ sin. $\frac{1}{2}$ (I + F) for the former, and $\frac{1}{2}\,i \div$ sin. $\frac{1}{2}$ (I−F) for the latter.

In figure 49, the radius b is infinite; the curve through B, and the chord, **BF**, coincide with the tangent **BI**; the deflection over the curves, and between the chords, is (F−I) and $\frac{1}{2}$ (F+I) for the right-hand crossing, and (F+I) and $\frac{1}{2}$ (F−I) for the left-hand.

It is also to be observed that the deflection F may be obtuse and equal to 180°−F′, or acute and equal to F′, and, for the same value of I, the radius of the circle AFB would be $\frac{1}{2}\,i \div$ cos. $\frac{1}{2}$ (I+F′) in the one case, and $\frac{1}{2}\,i \div$ sin. $\frac{1}{2}$ (I−F′) in the other.

There will be, therefore, four circles, with radii $\frac{1}{2}\,i \div$ sin. $\frac{1}{2}$ (I + F′), $\frac{1}{2}\,i \div$ sin. $\frac{1}{2}$ (I−F′), $\frac{1}{2}\,i \div$ cos. $\frac{1}{2}$ (I + F′), and $\frac{1}{2}\,i \div$ cos. $\frac{1}{2}$ (I−F′), on each of which the curves from the same tangent-lines will cross at the same angle, but with deflections differing in sign or magnitude.

3. *Demonstration of formulæ of problem XVI.* (*figure* 47).

These formulæ are based upon the relations between the sides and angles of the triangle AFB inscribed in the circle AFB, and the triangle AIB formed by the chord (**AB**) and the tangents through its extremities.

(**AB**) $= i$ is the distance from A to B; I and F are the deflections between the tangents crossing at I and F; Xa and Xb are the deflections over the arcs crossing at F; $k_a =$ (**AF**) and $k_b =$ (**FB**) are the chords, and a and b the radii of the arcs; $r =$ (**AO**) is the radius of the circle AFB; $o =$ (**MO**) is the perpendicular distance to its centre, O, from the middle of (**AB**); and A′ and B′ are the greater and lesser acute angles between **AB** and the tangents **AI** and **BI**. If the letters A and B are used without accents, they refer to the interior angles at the base of the triangle AIB when, as in the upper curves of figure 51, the arcs extend towards I from **AB**, and to the supplements of those angles when the arcs, as in the lower curves, extend in the opposite direction.

As **AI** and **BI** are the tangents to the arcs AF and BF, the

deflection from the tangents to the chords will be $\frac{1}{2}$ Xa and $\frac{1}{2}$ Xb, and the angles ABF and BAF, of the triangle ABF, are expressed in terms of A, B, Xa, Xb, by the following formulæ:

$$\left.\begin{array}{l} \text{BAF}= \left\{\begin{array}{l} \text{A}+\frac{1}{2}\,\text{X}a \text{ when the curve through A is convex to } \mathbf{AB}. \\ \text{A}-\frac{1}{2}\,\text{X}a \text{ when the curve through A is concave to } \mathbf{AB}. \end{array}\right. \\ \text{ABF}= \left\{\begin{array}{l} \text{B}+\frac{1}{2}\,\text{X}b \text{ when the curve through B is convex to } \mathbf{AB}. \\ \text{B}-\frac{1}{2}\,\text{X}b \text{ when the curve through B is concave to } \mathbf{AB}. \end{array}\right. \end{array}\right\} (1)$$

In figure 47, for instance, the arc AF_1 is convex, and $BAF=A+\frac{1}{2}Xa$; the arc AF_2 is concave, and $BAF_2=A-\frac{1}{2}Xa$.

In the lower pairs of curves, in figure 51, the angle ABF is equal to $180°-B-\frac{1}{2}Xb$ when the curve through B is concave, and to $180°-B+\frac{1}{2}Xb$ when it is convex.

As either angle of a triangle is the supplement of the sum of the others, the following equations are obtained by giving to BAF and ABF their values from (1), and representing AFB by K:

$$\left.\begin{array}{l} \text{BAF}=\text{A}+\frac{1}{2}\,\text{X}a=180°-(\text{K}+\text{B})-\frac{1}{2}\,\text{X}b. \\ \text{ABF}=\text{B}+\frac{1}{2}\,\text{X}b=180°-(\text{K}+\text{A})-\frac{1}{2}\,\text{X}a. \end{array}\right\} (2)$$

Representing, in these equations, $180°-(K+A)$ by Oa, and $180°-(K+B)$ by Ob, the values of BAF and ABF will be as follows:

$$\text{BAF}=Ob-\tfrac{1}{2}\,\text{X}b. \qquad \text{ABF}=Oa-\tfrac{1}{2}\,\text{X}a. \quad (2a)$$

As the central angle AOB, subtended by the chord, $(\mathbf{AB})=i$, of the circle AFB, is equal to (I+F), the values of r and o will be:

$$\left.\begin{array}{l} o=(\mathbf{MO})=\frac{1}{2}i.\ \text{cot.}\ \frac{1}{2}\,(\text{I}+\text{F}). \\ r=(\mathbf{OA})=\frac{1}{2}i\div\text{sin.}\ \frac{1}{2}\,(\text{I}+\text{F}). \end{array}\right\} (3)$$

In the inscribed triangle AFB, each side is a chord subtending the opposite angle, and $(\mathbf{AF})$ and $(\mathbf{BF})$ are also chords of arcs with radii, a and b, and as a chord is equal to twice its radius multiplied by the sine of the subtended angle,

$$\left.\begin{array}{l}(\mathbf{AF})=k_a=2.r.\sin.\ (\mathrm{B}+\frac{1}{2}\mathrm{X}b)=2.r.\sin.\ (\mathrm{O}a-\frac{1}{2}\mathrm{X}a)= \\ \qquad 2.a.\sin.\ \frac{1}{2}\mathrm{X}a. \\ (\mathbf{BF})=k_b=2.r.\sin.\ (\mathrm{A}+\frac{1}{2}\mathrm{X}a)=2.r.\sin.\ (\mathrm{O}b-\frac{1}{2}\mathrm{X}b)= \\ \qquad 2.r.\sin.\ \frac{1}{2}\mathrm{X}b.\end{array}\right\} \quad (4)$$

From these equations it follows that $r.\sin.\ (\mathrm{O}a-\frac{1}{2}\mathrm{X}a)=a.\sin.\ \frac{1}{2}\mathrm{X}a$.

Substituting, in this equation, $\sin.\ \mathrm{O}a\ .\ \cos.\ \frac{1}{2}\mathrm{X}a-\cos.\ \mathrm{O}a.\sin.\ \frac{1}{2}\mathrm{X}a$, for $\sin.\ (\mathrm{O}a-\frac{1}{2}\mathrm{X}a)$, and dividing by $\sin.\ \mathrm{O}a.\sin.\ \frac{1}{2}\mathrm{X}a$,

$$\text{Cot. } \tfrac{1}{2}\mathrm{X}a-\text{cot. } \mathrm{O}a=a\div r.\sin.\ \mathrm{O}a.$$

In the same manner it may be shown that,

$$\text{Cot. } \tfrac{1}{2}\mathrm{X}b-\text{cot. } \mathrm{O}b=b\div r.\sin.\ \mathrm{O}b.$$

$$\therefore \left\{\begin{array}{l}\text{Cot. } \frac{1}{2}\mathrm{X}a=a\div r.\sin.\ \mathrm{O}a+\text{cot. } \mathrm{O}a. \\ \text{Cot. } \frac{1}{2}\mathrm{X}b=b\div r.\sin.\ \mathrm{O}b+\text{cot. } \mathrm{O}b.\end{array}\right\} \quad (5)$$

Generally the first term of the second members of these equations will be positive, and the second term positive for convex, and negative for concave, curves; but there are exceptions to these rules, and to determine the signs in any particular case the magnitudes of the angles must be determined and the proper signs prefixed according to the rules given in proposition XIII., article 6. With regard to the method of determining these angles, see succeeding article.

When one of the angles, $\mathrm{X}a$ or $\mathrm{X}b$, has been obtained by (5), the other is determined by the following formula:

$$\text{Problem VIII. (1). } \mathrm{X}a+\mathrm{X}b=(\mathrm{I}-\mathrm{F}). \quad (6)$$

The radii, a and b, are determined by the following formulæ, derived from (4):

$$a=\left\{\begin{array}{l}r.\sin.\ (\mathrm{O}a-\frac{1}{2}\mathrm{X}a)\div\sin.\ \frac{1}{2}\mathrm{X}a=r.\sin.\ (\mathrm{B}+\frac{1}{2}\mathrm{X}b) \\ \qquad \div\sin.\ \frac{1}{2}\mathrm{X}a. \\ b.\sin.\ (\mathrm{O}a-\frac{1}{2}\mathrm{X}a).\sin.\ \frac{1}{2}\mathrm{X}b+\sin.\ (\mathrm{O}b-\frac{1}{2}\mathrm{X}b) \\ \qquad .\sin.\ \frac{1}{2}\mathrm{X}a.\end{array}\right\} \quad (7a)$$

$$b=\left\{\begin{array}{l}r.\sin.\ (\mathrm{O}b-\frac{1}{2}\mathrm{X}b)\div\sin.\ \frac{1}{2}\mathrm{X}b=r.\sin.\ (\mathrm{A}+\frac{1}{2}\mathrm{X}a) \\ \qquad \div\sin.\ \frac{1}{2}\mathrm{X}b. \\ a.\sin.\ (\mathrm{O}b-\frac{1}{2}\mathrm{X}b)\ .\ \sin.\ \frac{1}{2}\mathrm{X}a+\sin.\ (\mathrm{O}a-\frac{1}{2}\mathrm{X}a) \\ \qquad .\sin.\ \frac{1}{2}\mathrm{X}b.\end{array}\right\} \quad (7b)$$

In these equations, $(Oa-\frac{1}{2}Xa)=(B+\frac{1}{2}Xb)$, and $(Ob-\frac{1}{2}Xb)=(A+\frac{1}{2}Xa)$, are the angles ABF and BAF of (1) and (4).

4. *Angles Oa and Ob.*—The magnitudes of cot. Oa and cot. Ob, in (5), may be determined from the following considerations:

$$(2)\quad ABF=180°-(K+A)-\tfrac{1}{2}Xa=Oa-\tfrac{1}{2}Xa.$$

In this equation, making Xa equal to zero gives Oa as the value of the angle ABF. When $Xa=0$, the radius, a, is infinite, and the arc **AF** coincides with its tangent, **AI**, the intersection being on that line at the point Fa, figure 47. If, therefore, in any figure, a line be drawn from B to the point, Fa, at which the circle AFB cuts the tangent **AI**, Oa will be equal to the angle ABFa.

This angle, being subtended by the chord (**AF**a), is equal to half the central angle AOFa subtended by that chord, or to the angle which the tangents of the circle AFB makes with the chord at its extremities, A and Fa. In this case the arc coincides with its tangent, and Oa is therefore the angle between the tangent of the arc AF and the tangent of the circle AFB; and as every pair of arcs from the tangents through A and B will cross on the circle AFB at the same angle, whatever may be the radii, Oa is also the angle at which the tangent of the circle AFB crosses the tangent of the arc AF at every other point of intersection.

Oa may, however, be determined more readily as the angle between the chord **AF**a and the tangent of the circle AFB at their intersection, A.

The bearings, or deflections, of this tangent and chord from **AB** are $\frac{1}{2}(I+F)$ and A; and as, by proposition V., the deflection from one to the other is equal to the algebraic difference of their bearings, $Oa=A-\frac{1}{2}(I+F)$. In figure 47, $I=+(A'-B')$; $F=+F'$; $\frac{1}{2}(I+F)=\frac{1}{2}(A'-B'+F')$; $A=180°-A'$; and as the chord and tangent are on different sides of **AB**, $Oa=A+\frac{1}{2}(I+F)=180°-A'+\frac{1}{2}(A'-B'+F')=180°-\frac{1}{2}(A'+B'-F')$. The value of Ob, obtained by the same process, will be as follows:

$$Ob=B+\tfrac{1}{2}(I+F)=B'+\tfrac{1}{2}(A'-B'+F')=\tfrac{1}{2}(A'+B'+F').$$

It will be seen that Ob differs from $\frac{1}{2}(I+F)$ in the signs of B′ and F′, and from Oa in the sign of F′.

This rule is general, whatever may be the values of I and F, and is expressed in formulæ (6), (7), and (8), problem XVI.

These formulæ do not, however, determine whether $\frac{1}{2}(A' \pm B' \pm F')$ is the angle O*a* or O*b*, or the supplement of O*a* or O*b*, and do not, therefore, determine the signs of cot. $\frac{1}{2}$ O*a* and cot. $\frac{1}{2}$ O*b* in (5).

In table III., problem XVI., the signs and values of I, F, cot. O*a*, and cot. O*b* are given for the different forms of curves in the figures to which reference is made. It will be seen that the difficulty of application is in the determination of the signs of the cotangents of O*a* and O*b* in the formulæ for X*a* and X*b*.

PROPOSITION XV.

TURN-OUTS, SIDINGS, AND CROSSINGS FROM RAILROAD-TRACKS.

(Demonstration of formulæ of problem XVIII.)

1. These formulæ are the same with those of problem XVI., except that the symbols used to represent some of the quantities are different.

Turn-outs from Switches (figure 59). g is the gauge; s the throw; S the switch angle; a and Da the radius and degree of curvature of the main-track rail crossed by the turn-out rail; b the radius, and Db the degree, of the turn-out rail; A is on the main track, and B on the turn-out rail, at end of switch; $(\mathbf{AB})=i=(g-s)$ is the distance from A to B; $A'=90°$, and $B'=(90°-S)$, are the acute angles between **AB** and the tangents **AI** and **BI**, the former parallel, and the latter at an angle, S, to the main-track tangent AI, from the end, or toe, of the switch; X*a* and X*b* are the tangential deflections over the crossing arcs, AF and FB. and $k_a=(\mathbf{AF})$ and $k_b=(\mathbf{BF})$ are the chords, or frog-distances, from A and B to the crossing, F, of the two arcs; I=S is the deflection between the tangents crossing at I, and F, called the frog-angle, is the angle between the tangents crossing at F. O is the centre, and r the radius, of the circle AFB, and o the distance to O, at right angles to **(AB)**, from its middle point.

The turn-out is to the inside when to the concavity of the

main track, as for the curves crossing at F_1, and to the outside when to the convexity, as for the curves crossing at F_2 and F_3. When the turn-out is to the inside, AF_1 is convex, and BF_1 concave, to **AB**; and when to the outside, AF_2 or AF_3 is concave, and BF_2 concave and BF_3 convex, to **AB**.

The formulæ of table III., problem XVI., apply to turn-outs, by substituting 90° for A′, 90°−S for B′, $(g-s)$ for i, and for I the switch-angle, S, when the turn-out is to the inside, as in figures 51¼ and 59¼, and 180°−S when the turn-out is to the outside, as in figures 53 and 59½.

$n=\frac{1}{2}\cot.\ \frac{1}{2}\,\mathrm{F}$ is the frog-number for an angle F. (1)

$o=\frac{1}{2}(g-s).\cot.\ \frac{1}{2}(\mathrm{F}+\mathrm{S})$=distance from middle of (**AB**) to centre of circle ABF. (2)

$r=\frac{1}{2}(g-s)\div\sin.\ \frac{1}{2}(\mathrm{F}+\mathrm{S})=o\times\cos.\ \frac{1}{2}(\mathrm{F}+\mathrm{S})$=radius of circle. (3)

Case 1. *When a is infinite, and the main track, therefore, straight.*

In this case, for distinction from a curved main track, let (**AF***s*) and (**BF***s*) be the frog-distances, and b_s and D*s* the radius and degree of the turn-out curve.

$$\left.\begin{aligned}(\mathbf{AF}s)&=2\,.\,o=(\mathbf{BF}s)\,.\,\cos.\ \tfrac{1}{2}(\mathrm{F}+\mathrm{S}).\\(\mathbf{BF}s)&=2\,.\,r=(\mathbf{AF}s)\div\cos.\ \tfrac{1}{2}(\mathrm{F}+\mathrm{S}).\end{aligned}\right\}\ (4)$$

$$\left.\begin{aligned}b_s&=r\div\sin.\ \tfrac{1}{2}(\mathrm{F}-\mathrm{S})=(g-s)\div(\cos.\ \mathrm{S}-\cos.\ \mathrm{F})=(\mathbf{AF}s)\\&\quad\div(\sin.\ \mathrm{F}-\sin.\ \mathrm{S}).\\\text{Sin. }\tfrac{1}{2}\mathrm{D}s&=50\,.\,\sin.\ \tfrac{1}{2}(\mathrm{F}-\mathrm{S})\div r=50\,.\,(\cos.\ \mathrm{S}-\cos.\ \mathrm{F})\\&\quad\div(g-s).\end{aligned}\right\}\ (5)$$

$$\left.\begin{aligned}\text{Cos. F}&=\cos.\ \mathrm{S}-(g-s)\div b_s=\cos.\ \mathrm{S}-2\,.\,(g-s)\,.\,\sin.\ \tfrac{1}{2}\,\mathrm{D}s\\&\quad\div 100.\\\text{Sin. F}&=\sin.\ \mathrm{S}-2\,.\,o\div b_s=\sin.\ \mathrm{S}-(\mathbf{AF}s)\,.\,\sin.\ \tfrac{1}{2}\,Ds\\&\quad\div 100.\end{aligned}\right\}\ (6)$$

Case 2. *When the main-track rail,* AF, *is curved.*

The turn-out is to the inside when AF is convex to **AB**.
" " outside " concave "

When a double sign is prefixed to a term of the following equations, use the upper sign when the turn-out is to the inside, and the lower sign when it is to the outside.

$$\text{Cot. } \tfrac{1}{2}\,\text{X}a = [a \pm \tfrac{1}{2}(g-s)] \div o. \quad (7)$$

$$\tfrac{1}{2}\,\text{X}b = \tfrac{1}{2}(\text{F}-\text{S}) \pm \tfrac{1}{2}\,\text{X}a. \quad (8)$$

$$b = \frac{r\,.\,\text{cos. } \tfrac{1}{2}\,\text{X}a}{\text{sin. } \tfrac{1}{2}\,\text{X}b} = \frac{a \pm \tfrac{1}{2}(g-s)}{\dfrac{a\,.\,(\text{cos. S}-\text{cos. F})}{(g-s)} \pm \text{cos. S}}. \quad (9)$$

$$(\mathbf{AF}) = k_a = 2\,.\,a\,.\,\text{sin. } \tfrac{1}{2}\,\text{X}a = (\mathbf{BF}s)\,.\,\text{cos. } (\text{S} + \tfrac{1}{2}\,\text{X}b).$$
$$(\mathbf{BF}) = k_b = 2\,.\,b\,.\,\text{sin. } \tfrac{1}{2}\,\text{X}b = (\mathbf{BF}s)\,.\,\text{cos. } \tfrac{1}{2}\,\text{X}a. \quad (10)$$

$$\text{Cos. F} = \begin{cases} \text{cos. S} - \dfrac{(g-s)\,.\,[a \pm b\,.\,\text{cos. S} \pm \tfrac{1}{2}(g-s)]}{a\,.\,b} \\ \text{cos. S} - \dfrac{2\,.\,(g-s)\,.\,(\text{sin. } \tfrac{1}{2}\,\text{D}b\,.\,\text{cos. S} \pm \text{sin. } \tfrac{1}{2}\,\text{D}a)}{100} \mp \dfrac{\tfrac{1}{2}(g-s)^2}{a\,.\,b} \end{cases} \quad (11)$$

In (7), $a \pm \frac{1}{2}(g-s)$ is the radius of a curve through the middle of (**AB**), differing only by $\frac{1}{2}s$ from the radius of the centre line of the main track, and cot. $\frac{1}{2}$Xa may, therefore, be assumed to vary directly with the radius, and inversely with the degree, of the curve of the centre line of the main track.

In (8), when the turn-out is to the outside, the turn-out curve, BF_2 (figure 59), will be concave to **AB** when (F—S) is greater than Xa, and convex when it is less, as for the curve BF_3.

In (10), (**BF**s) = 2 . r is the frog-distance for a straight track, calculated by (4).

2. *Three-throw, or Double, Switches* (figure 68).—When turn-outs are made from the same switch to both sides of a track, the frog at the intersection of the turn-out rails is called the middle frog, and the switch a three-throw, or double, switch.

In this case, A″ and B″, each at a distance s from the main-track rail, are the junctions of the switch and turn-out rails; (**A″B″**) = i = (g − 2s) is the distance from A″ to B″; A″ = B″ = 90° − S; I″ = A″ + B″ = 180° − 2 S; F = 180° − F′, and the equations are those of figure 56, table III., problem XVI. In the formulæ, F″ will represent the middle frog-angle, and Fa and Fb the frog-angles on the main track; O″ and r'' the centre and radius of the circle A″B″F″, and o'' the perpendicular distance to its centre from the middle of (**A″B″**).

$$o'' = \tfrac{1}{2}(g-2s)\,.\,\text{cot. } \tfrac{1}{2}(\text{F}''+2\,\text{S}). \quad (12)$$
$$r'' = \tfrac{1}{2}(g-2s) \div \text{sin. } \tfrac{1}{2}(\text{F}''+2\,\text{S}). \quad (13)$$

Case 1. *The main track straight;* $a''=b''$, and $Fa = Fb = F$.

In this case, F″ may be determined in terms of $(g-2s)$ and F, or in terms of $(g-2s)$ and the main-track frog-distance **AF***s*), as follows:

$$\text{Cos. } \tfrac{1}{2} F'' = \left\{ \begin{array}{l} \text{cos. S} - \dfrac{\frac{1}{2}(g-2s) \,.\, (\text{cos. S} - \text{cos. F})}{(g-s)} \\ \\ \text{cos. S} - \dfrac{\frac{1}{2}(g-2s) \,.\, (\text{sin. F} - \text{sin. S})}{(\mathbf{AF}s)} \end{array} \right\} \quad (14)$$

The middle frog will be in the centre of the track, at a distance k'' from A″ and B″, determined by the following formula:

$$k'' = \tfrac{1}{2}(g-2s) \div \text{sin. } \tfrac{1}{2}(\tfrac{1}{2} F'' + S). \quad (15)$$

Case 2. *When the degrees,* D″*a and* D″*b, of the turn-out curves and the main-track frog-angles are different.*

$$\text{Cot. } \tfrac{1}{2} Xa = a'' \div r.\text{cos. } \tfrac{1}{2} F'' - \text{tan. } \tfrac{1}{2} F''. \quad (16)$$

$$\tfrac{1}{2} X''b = \tfrac{1}{2}(F'' - 2S) - \tfrac{1}{2} X''a. \quad (17)$$

$$\left. \begin{array}{l} k''_a = 2.r.\text{cos. } \tfrac{1}{2}(F'' - X''a). \\ k''_b = 2.r.\text{cos. } \tfrac{1}{2}(F'' - X''b). \end{array} \right\} \quad (18)$$

$$\text{Cos. } F'' = \left\{ \begin{array}{l} \text{cos. } 2S - \dfrac{(g-2s).[\text{cos. S}.(a+b) - \frac{1}{2}(g-2s)]}{a.b} \\ \\ \text{cos. } 2S - \dfrac{2.(g-2s).\text{cos. S}.(\text{sin. } \frac{1}{2} D''a + \text{sin. } \frac{1}{2} D''b)}{100} + \dfrac{\frac{1}{2}(g-2s)^2}{a.b} \end{array} \right\} \quad (19)$$

To obtain the position of the middle frog, one of the turn-out curves must first be laid off, and the distance k''_a or k''_b measured on that curve. As a rule the first step will be to determine the middle frog-angle by (19), and then the angle $\frac{1}{2} Xa$ or $\frac{1}{2} Xb$ by (16). If the radius b'' is given, determine X″b by substituting b'' for a''.

3. *Sidings* (figure 50).—In this case, the turn-out curve from the frog-tangent at F is produced until parallel to the main track, and, as the tangents at A and B are parallel, A′ = B′ = 90°, and I=0. If the distance between the centres of the siding and main track be represented by c, $i = (\mathbf{AB}) = (c-g)$, and the formulæ will be the same with those for turn-outs, except that S is equal to zero. The siding will be on the inside of the main track, as in figure 50¼, or on the outside, as in figures 50½ and 50¾.

When on the inside, the main-track rail, AF_1, will be convex, and the turn-out rail, BF_1, concave. When the siding is on the outside of the main track, AF_2 or AF_3 will be concave to **AB**, and BF_2 will be concave and BF_3 convex.

The frog-number, n, or the angle, F; the radius, a, or the degree, Da, of the main-track rail, AF; and the distance, $i = (c-g)$, between the intersecting rails, are given, and it is required to determine the radius, r, of the circle AFB; the perpendicular distance, o, to its centre, O, from the middle of (**AB**); the deflections, Xa and Xb, over the curves AF and FB; the chords, or frog-distances, $k_a = (\mathbf{AF})$, and $k_b = (\mathbf{BF})$, and the radius, b, of the turn-out curve.

$$o = \tfrac{1}{2}(c-g) . \cot. \tfrac{1}{2} F = (c-g).n. \quad (20)$$
$$r = \tfrac{1}{2}(c-g) \div \sin. \tfrac{1}{2} F = o \div \cos. \tfrac{1}{2} F. \quad (21)$$

Case 1. *When a is infinite, and the main track, therefore, straight.*

To distinguish from the quantities for a curved track, let the frog-distances be represented by (**AF**s) and (**BF**s), and the degree and radius of the turn-out curve by b_s and Ds.

$$X a = 0, \; X b = F. \quad (22)$$
$$k_a = (\mathbf{AF}s) = 2.o. \quad k_b = (\mathbf{BF}s) = 2.r. \quad (23)$$
$$\left.\begin{array}{l} b_s = r \div \sin. \tfrac{1}{2} F = (c-g) \div \text{vers. F.} \\ \text{Sin. } \tfrac{1}{2} Ds = 50 . \sin. \tfrac{1}{2} F \div r = 100 . \text{vers. F} \div 2 . (c-g). \end{array}\right\} \quad (24)$$

Case 2. *When the main track is curved.*

The siding will be on the inside when the main track is convex to (**AB**), and on the outside when it is concave.

When a double sign is prefixed to any term in the equations below, use the upper sign when the turn-out is to the inside, and the lower sign when it is to the outside.

$$\text{Cot. } \tfrac{1}{2} Xa = [a \pm \tfrac{1}{2}(c-g)] \div o = [a \pm \tfrac{1}{2}(c-g)] \div (c-g).n. \quad (25)$$
$$\tfrac{1}{2} Xb = \tfrac{1}{2} F \pm \tfrac{1}{2} Xa. \quad (26)$$
$$\left.\begin{array}{l} k_a = (\mathbf{AF}) = 2 . a . \sin. \tfrac{1}{2} Xa = 2 . r . \cos. \tfrac{1}{2} Xb = \\ \qquad (\mathbf{BF}s) . \cos. \tfrac{1}{2} Xb. \\ k_b = (\mathbf{BF}) = 2 . b . \sin. \tfrac{1}{2} Xb = 2 . r . \cos. \tfrac{1}{2} Xa = \\ \qquad (\mathbf{BF}s) . \cos. \tfrac{1}{2} Xa. \end{array}\right\} \quad (27)$$
$$b = r . \cos. \tfrac{1}{2} Xa \div \sin. \tfrac{1}{2} Xb = a . \sin. Xa \div \sin. Xb. \quad (28)$$

In (25), $a \pm \tfrac{1}{2}(c-g)$ is the radius of a curve through the

middle of (**AB**). In (26), when the siding is on the inside, and X*a* is greater than F, the turn-out curve is convex to **AB**, as in figure 50.

4. *Crossings from one track to another* (figure 70) are made between turn-out curves from switches on the two tracks, to which the curve of the crossing is intermediate, and with one of which it is reversed. If the crossing is straight, it is a tangent between the turn-out curves. Let F*a* be the greater, and F*b* the lesser, of the frog-angles, and let B be a point on the inner rail of the crossing, at a distance from F*b*, measured at right angles to the frog-tangent, equal to the gauge, *g*, of the track. Let A be a point on the main track opposite F*b*, such that a radial line through A will pass through B. The curves, AF*a* and BF*a*, intersect at F*a*; the angle A′, between the tangent at A and **AB**, is 90° ; the angle B′, between the tangent at B and **AB**, is equal to 90°−F*b*; the chord, or frog-distance, (**AF***a*), is the distance between the frogs, measured on the parallel tracks ; and if *c* is the distance between the centres of those tracks, the distance (**AB**) = *i* is equal to $c-g\,.\,(1+\cos.\ \mathrm{F}b)$, which may, without appreciable error, be assumed equal to $(c-2.g)$.

The curves and angles are, therefore, in all respects the same with the curves of a turn-out to the inside from a switch, with a switch-angle S = F*b*, and with the distance (**AB**) = $(g-s)=(c-2.g)$, and the formulæ for crossings may therefore be obtained from the formulæ for turn-outs by substituting F*b* for S, and $(c-2.g)$ for $(g-s)$.

Case 1. *When a is infinite and the parallel tracks straight.*

X*a*=0. X*b*=(F*a*−F*b*). (29)

$$\left.\begin{aligned} k_a &= (\mathbf{AF}a) = (c-2\,.\,g)\,.\,\text{cot.}\ \tfrac{1}{2}\,(\mathrm{F}a+\mathrm{F}b) = 2\,.\,o. \\ k_b &= (\mathbf{BF}b) = (c-2\,.\,g) \div \text{sin.}\ \tfrac{1}{2}\,(\mathrm{F}a+\mathrm{F}b) = 2\,.\,r. \end{aligned}\right\} \quad (30)$$

$$b = (c-2\,.\,g) \div (\cos.\ \mathrm{F}b - \cos.\ \mathrm{F}a). \quad (31)$$

When the frog-angles are equal, X*b* = 0, *b* is infinite, and the crossing straight, the frog-distance being as follows:

$k_a = (\mathbf{AF}a) = (c-2\,.\,g)\,.\,\text{cot.}\ \mathrm{F}a.$ (32).

Case 2. *When the parallel tracks are curved.*

$$\text{Cot.}\ \tfrac{1}{2}\,\mathrm{X}a = a \pm \tfrac{1}{2}\,(c-2\,.\,g) \div o. \quad (33)$$

$$\tfrac{1}{2}\,\mathrm{X}b = \tfrac{1}{2}\,(\mathrm{F}a-\mathrm{F}b) + \tfrac{1}{2}\,\mathrm{X}a. \quad (34)$$

$$b = \frac{r \,.\, \text{cos. } \frac{1}{2}\,\text{X}a}{\text{sin. } \frac{1}{2}\,\text{X}b} = \frac{a + \frac{1}{2}(c - 2\,.\,g)}{\frac{a\,.\,(\text{cos. F}b - \text{cos. F}a)}{(c - 2\,.\,g)} + \text{cos. F}b} \quad (35)$$

$$k_a = (\mathbf{A}\mathbf{F}a) = 2\,.\,a\,.\,\text{sin. } \tfrac{1}{2}\,\text{X}a = 2\,.\,r\,.\,\text{cos. } (\text{F}b + \tfrac{1}{2}\,\text{X}b)$$
$$k_b = (\mathbf{B}\mathbf{F}a) = 2\,.\,r\,.\,\text{cos. } \tfrac{1}{2}\,\text{X}a = 2\,.\,b\,.\,\text{sin. } \tfrac{1}{2}\,\text{X}b. \quad (36)$$

5. *Approximate Formulæ for Turn-outs, Sidings, and Crossings.*—All problems relating to turn-outs, sidings, and crossings may be solved, without appreciable error, by approximate formulæ based on the following propositions:

The *rate of curvature* of a circular arc is constant, and equal to the tangential deflection over the arc, divided by its length. The *degree of curvature* is the deflection over an arc of which the chord is 100 feet, and as, in such case, for all curves used in railroad alignments, the arc may be assumed equal to the chord, the rate of curvature, or the curvature in one foot, is equal to the degree of curvature divided by 100, and the total curvature to the rate of curvature multiplied by the length of the arc.

If, therefore, Da, Xa, and Ia is the degree, curvature, and length of one arc, and Db, Xb, and Ib of another arc,

$$\text{X}a = \text{D}a\,.\,\text{I}a \div 100. \quad \text{X}b = \text{D}b\,.\,\text{I}b \div 100.$$
$$\text{X}a + \text{X}b = (\text{D}a\,.\,\text{I}a + \text{D}b\,.\,\text{I}b) \div 100. \quad \text{(a)}$$

Assuming, for arcs not exceeding one or two hundred feet in length, that the arc is equal to the chord, and that the arcs in (a) are of equal length, if k is the common chord length of the two arcs,

$$\text{D}a + \text{D}b = 100\,.\,(\text{X}a + \text{X}b) \div k. \quad (37)$$

For small angles it may also be assumed that the angles vary in direct proportion to their sines, and, therefore, the sine of the half-angle, in terms of the angle, may be expressed by the following equation:

$$\text{Sin. } \tfrac{1}{2}\,\text{I} = \text{I} \div (114.4 + .04.1). \quad (38)$$

The degree of approximation to the exact values will be seen by the following examples:

Sin. ($\frac{1}{2}10^\circ = 5^\circ$) $= 10 \div (114.4 + .4) = .0871$. Correct value, .0871.
Sin. ($\frac{1}{2}20^\circ = 10^\circ$) $= 20 \div (114.4 + .8) = .1736$. Correct value, .1736.

Sin. ($\frac{1}{2}$ 30° = 15°) = 30 ÷ (114.4 + 1.20) = .2595. Correct value, .2588.

It has been shown, in proposition XIV., that every pair of curves from the same tangents, at A and B, and intersecting at the same angle, F, will cross on the circumference of the same circle AFB, of which $\frac{1}{2}i \div$ sin. $\frac{1}{2}$ (F + I) is the radius, and it may be shown, for small values of i, and for the classes of curves under consideration, that the chords are nearly equal to each other and to the arcs which they subtend. The maximum value for any chord is the diameter, $2 \cdot r$, of the circle AFB, from which the minimum value may differ, in extreme cases, a half of a foot, corresponding to a deflection, on a ten-degree curve, of three minutes.

It may, therefore, be assumed, without appreciable error, for each pair of curves from the same tangent-points at A and B, and intersecting at the same angle, F, that the chords, (**AF**) and (**BF**), when AF is a curve, are equal to each other and to the chords, (**AF**s) and (**BF**s), for a straight main track, with radius, a, equal to infinity.

If, therefore, F and I are the deflections between the pairs of tangents crossing at F and I; $i = (\mathbf{AB})$, the distance from A to B; Xa and Xb the tangential deflections over the arcs AF and BF, and (**BF**) = k the frog-distance from B or A,

Problem VI. (1). Xa + Xb = (F − I). (c)
Proposition XIV. (4). (**BF**s) = k = i. sin. $\frac{1}{2}$ (F + I). (d)

Substituting in (d) the value of sin. $\frac{1}{2}$ (F + I), from (38), and in (37) the value of (Xa + Xb) from (c), and of k from (39),

(**BF**s) = $k = i \cdot [114.4 \div (\mathrm{I} + \mathrm{F}) + .04] = i \cdot 114.4 \div (\mathrm{I} + \mathrm{F}) +$ (39)
(Db + Da) = .874. (F² − I²) ÷ i = 100 × (F − I) ÷ k. (40)

As, for the same values of F and I, the algebraic sum, Da + Db, will be constant for all pairs of curves, it will be equal to the degree, Ds, of the turn-out curve when the main track is straight, and Da is, therefore, equal to zero, and may be determined as follows:

Proposition XV. (5). Sin. $\frac{1}{2}$ Ds = 50 . (cos. I − cos. F) ÷ i.

On a standard gauge of 4.71, for example, with $i = (g - s) =$ 4.29, the switch-angle, S = I = 1° 00′, and F = 8° 10′, the de-

gree, Ds, of the turn-out curve, when Da=0, is 13° 22′ by the exact formula, and 13° 23′ by the approximate formula.

It should be stated that the sum (Da+Db) is algebraic, and the signs are to be determined by the rules of problem XVIII., table IX., to which reference is made for application of the formulæ to turn-outs, sidings, and crossings.

These formulæ may be applied, by making F=0, to compound and reversed curves, when the lengths of the arcs are nearly equal, as exemplified in problem XXII., example 6.

PROPOSITION XVI.

(Demonstration of problems XIX. to XXVII.)

In all of these problems A and B are tangent-points on two curves ; (**AB**)=i is the distance from A to B ; the tangents through A and B intersect at a point I ; A′ and B′ are the greater and lesser acute angles between those tangents and the line **AB** ; a and b are the radii of the curves, and t the intermediate common tangent ; Ia and Ib are the tangential deflections over the two arcs, and I=Ia+Ib= (A′ ± B′), or 180°− (A′ ± B′), is the angle of intersection of the tangents intersecting at I, determined by the rules of problem V. ; C and Z are the centres of the curves, and (**CZ**)=c is the distance from C to Z.

2. (Figures 93 and 94.) The formulæ are obtained by the method of latitudes and departures explained in problem XI. The origin of co-ordinates, C, is assumed to be at the centre of one of the curves (best at the centre of the curve of longer radius).

In this proposition it is assumed to be the centre of the curve through A, corresponding to the greater of the two angles, A′ and B′.

The line (**CZ**)=c is the closing line to the three lines (**CA**)=a, (**AB**) = i, and (**BZ**)=b, observing that the direction CA is *from* the centre *to* A, and the direction BZ *from* B *to* the centre Z.

The co-ordinate axis, **CN**, which will be designated as north, is from C towards, and at right angles to, **AB**, so that the bearing CN.AB is equal to 90°. Looking from C towards **AB**, the right will be positive, for bearings and departures, when B is to the right of A, and the left when B is to the left. In figure 93, B is to the right of A; and if CN were north, east bearings would be positive, and left bearings negative.

3. *Bearings of* (**CA**), (**AB**), *and* (**BZ**).—The curve through A or B is convex or concave to **AB** when, if extended towards the opposite tangent-point B or A, the convexity or concavity is towards **AB**.

In figure 93, both curves are concave; in figure 94, the curve through A is convex, and the curve through B concave. In figure 86, both curves are concave, because, although the arc BT is convex when extended away from A, it would be concave when extended towards A.

The bearings of the radii will be positive when the curves are convex, and negative when they are concave.

In figure 93, for instance, CN is north, and east bearings positive; CN.CA is northwest, and CN.BZ southwest, both curves being concave. In figure 94, the curve through A is convex, and the curve through B concave; CA bears northeast, and BZ northwest. The bearing CN.AB is, by supposition, always + 90°.

As **CN** and **CA** are perpendicular to **AB** and **AI**, the bearing CN.CA will be equal to the angle A′ between **AB** and **AI**, or to 180°−A′.

For similar reasons the bearing of **BZ** is equal to B′ or to 180°−B′.

As C is the origin of co-ordinates, the bearing CN.CA will be always north and equal to the acute angle A′. The bearing CN.BZ will also be to the north and equal to B′ when the centres are on different sides of **AB**, but when on the same side, as in figure 93, the bearing will be to the south and equal to 180°−B′.

The bearing of either radius will, therefore, be positive or negative, as the curve may be convex or concave to **AB**; the bearing of BZ will be obtuse or acute, as the centres may be on the same or different sides of **AB**, and the bearing of CA will be always acute, or less than 90°.

In the application of these rules, when the origin, C, is at

the centre of the curve through B, substitute A for B, and B for A.

From these considerations, the bearings CN.CA = A, and CN.BZ=B, in terms of the acute angles, A′ and B′, will be as follows:

A=CN.CA=A′ is always acute. (1)

B=CN.BZ= B′ for centres on different sides of **AB.**
" " =(180°−B′) " " the same side " } (2)

Either bearing will be plus when the curve is convex to **AB.**
" " " minus " " concave "

Observe that when B is negative and equal to −(180°−B′)= −180°+B′, it is not equal to +180°−B′. Thus, for instance, if CN is north and B′ equal to 20°, CN.BZ= −(180°−B′)= −180°+B′= −160° is N. 160° W. or S. 20° W., while +180° −B′= +160° is N. 160° E. or S. 20° E.

4. *Problem XXIV.*—In this problem, *a*, *b*, *i*, A′, and B′ are given, and also whether the curves are convex or concave, and whether the centres are on the same or different sides of **AB,** and it is required to determine the bearing CN.CZ=C, and the length, (**CZ**)=*c*, of the line from C to Z, and also the radial deflections, CA.CZ and CZ.BZ, from CA to CZ and from CZ to BZ.

Considering (**CZ**) as the closing line to (**CA**), (**AB**), and (**BZ**), determine the bearings A and B by (1) and (2), noting carefully the signs as well as the magnitudes.

From the bearings and lengths of *a*, *i*, and *b*, the bearing, C, and length, *c*, of the closing line, (**CZ**), may be determined by (2), problem XI.

The deflection from any one of the three directions, CA, CZ, and BZ, to another is then calculated as the algebraic difference of their bearings from CN, according to the rules of problem III. and proposition V.

Representing by Σd and Σl the departure and latitude of the closing line (**CZ**),

$$\Sigma d = +i \pm a.\ \text{sin. A}' \pm b.\ \text{sin. B}'. \quad (3)$$
$$\Sigma l = +a.\ \text{cos. A}' \pm b.\ \text{cos. B}'. \quad (4)$$
$$\text{Tan. C} = \Sigma d \div \Sigma l. \quad (5)$$
$$c = \Sigma d \div \text{sin. C} = \Sigma l \div \text{cos. C}. \quad (6)$$

C and c may also be calculated as follows:

$$c^2=(\Sigma d)^2+(\Sigma l)^2. \quad (7)$$

$$\text{Sin. C}=\Sigma d \div c. \quad \text{Cos. C}=\Sigma l \div c. \quad (8)$$

$$\text{CA.CZ}=\text{CN.CZ}-\text{CN.CA}=\text{C}-\text{A}. \quad (9)$$

$$\text{CZ.BZ}=\text{CN.BZ}-\text{CN.CZ}=\text{B}-\text{C}. \quad (10)$$

$$\text{CA.BZ}=\text{R}=\text{CN.BZ}-\text{CN.CA}=\text{B}-\text{A}. \quad (11)$$

In (3), the departure, a. sin. A′ or b. sin. B′, is plus when the curve through A or B is convex, and minus when it is concave.

In (4), the latitude, b.cos. B′, is plus when the centres are on different sides of **AB**, and minus when they are on the same side.

The signs of Σd and Σl are to be carefully noted.

In (5) and (8), C will have the same sign with Σd, and if C′ is the acute angle corresponding to the numerical value of the tangent, sine, or cosine, C is equal to C′ when Σl is positive, and to 180° − C′ when Σl is negative.

In (9), (10), and (11), the differences are algebraic, and the signs of A, B, and C are to be noted, as well as their magnitudes.

If the calculated value of any bearing exceeds 180°, subtract from 360° and change the sign, according to the rules of problem II.

These deflections are radial, or from one radial direction to another, and not necessarily equal to the tangential deflections over the arcs between the extremities of the radii, as explained in the succeeding article.

5. *Problem XXV.*—In this problem, the data are the same as in problem XXIV., but the curves are supposed to be connected by one or more pairs of tangents, of which the lengths and positions are to be determined. The relations between the lengths of these tangents and the distance, c, from C to Z, are determined in the following manner:

(Figures 90, 91, and 92.) C and Z are the centres of the curves; t is the length, and Ta and Tb the extremities, of the common tangent to the two curves; T=CZ.CTa is the deflection from CZ to the radial direction CTa. The curves are reversed when, as in figure 90, the centres are on different sides of **T**a**T**b, and compound when, as in figures 91 and 92, they are on the same side.

Assuming the axis CN to coincide with the radius $\mathbf{CT}a$, $\mathrm{CN}.\mathrm{CT}a = 0$; $\mathrm{CN}.\mathrm{T}a\mathrm{T}b = +90°$; $\mathrm{CN}.\mathrm{T}b\mathrm{Z} = 0$ for reversed curves, and 180° for compound curves. Considering $(\mathbf{CZ}) = c$ as the closing line to $(\mathbf{CT}a) = a$, $(\mathbf{T}a\mathbf{T}b) = t$, and $(\mathbf{T}b\mathbf{Z}) = b$, the departure, d; the latitude, l; the length, c, and the bearing, T, may be calculated by the formulæ of the preceding article as follows:

(3) $d = t$. sin. $90° \pm a$. sin. $0 \pm b$. sin. $0 = t$. (12)

(4) $l = \left\{ \begin{array}{l} t \text{ . cos. } 90° + a \text{ . cos. } 0 + b \text{ . cos. } 0 = (a+b) \text{ for reversed curves.} \\ t \text{ . cos. } 90° + a \text{ . cos. } 0 - b \text{ . cos. } 180° = (a-b) \text{ for compound curves.} \end{array} \right\}$ (13)

(5) Tan. $\mathrm{T} = d \div l = t \div (a+b)$. (14)

(6) $c = t \div$ sin. $\mathrm{T} = (a \pm b) \div$ cos. T. (15)

T and c may also be determined as follows:

(7) $c^2 = t^2 + (a \pm b)^2$. (16)

(8) Sin. $\mathrm{T} = t \div c$. Cos. $\mathrm{T} = (a \pm b) \div c$. (17)

Substituting in (15) and (17) the value of c from (6),

Cos. $\mathrm{T} = (a \pm b)$. sin. $\mathrm{C} \div \Sigma d = (a \pm b)$. cos. $\mathrm{C} \div \Sigma l$. (18)

$t = \Sigma d$. sin. $\mathrm{T} \div$ sin. $\mathrm{C} = \Sigma l$. sin. $\mathrm{T} \div$ cos. C. (19)

In (13), substitute for $(a \pm b)$, $(a-b)$ when the curves are compound, and $(a+b)$ when they are reversed.

In (14), T is acute and equal to T′ for reversed curves, and also for compound curves when a is greater than b, and $(a-b)$ is therefore positive; but when the curves are compound, and b is greater than a, $(a-b)$ is negative, and T is equal to 180° − T′, and obtuse.

For this reason, when dealing with compound curves, take the origin, C, at the centre of the curve with the longer radius, in order to avoid the use of obtuse angles.

As $(\mathbf{CZ}) = c$ is the hypothenuse of a right-angled triangle, it is evident that, in figures 90, 91, or 92, a similar triangle may be constructed on the opposite side of $(\mathbf{CZ})$, and, therefore, the following conclusions may be deduced from the consideration of the two equal triangles, and from the equation, $c^2 = t^2 + (a \pm b)^2$.

When C is less than $(a-b)$, as in figure 95, the circles are one within the other, and a common tangent is not possible.

When c is equal to $(a-b)$, as in figure 96, $t=0$, and the curves touch internally at a point of compound curvature.

When c is less than $(a+b)$, and greater than $(a-b)$, as in figure 97, the curves intersect, and there will be one pair of equal tangents, to which the curves will be compound.

When $c=(a+b)$, as in figure 98, $t=0$; the curves will touch externally at a point of reversed curvature, and there will also be a pair of tangents, to which the curves will be compound.

When c is greater than $(a+b)$, as in figure 97, there will be two pairs of equal tangents, to one of which the curves will be reversed and compound to the other.

Let Ta and T'a, and Tb and T'b, figure 101, be the extremities of the pair of equal tangents at intersection with the radii from C and Z. The two radii, **CT**a and **CT**'a, will be on opposite sides of **CZ**, and CZ . CTa = CZ . CT'a = T. As CN . CZ = C is the bearing of CZ, CN . CTa = (C−T), and CN . CT'a = (C+T), are the bearings of **CT**a and **CT**'a.

When (C ± T) is a term of any of the following formulæ, it is to be understood that (C−T) is the bearing of one of these radii, and (C+T) of the other.

If, then, the bearings, CN . CA = A, CN . BZ = B, and CN . CZ = C, be determined by the formulæ of the preceding article, and T and t by (18) and (19), the tangential deflection Ia will be equal to the algebraic difference of the bearings of CTa and CA, and Ib to the algebraic difference of the bearings of TbZ and BZ.

The formulæ will be as follows :

(3) $\Sigma d = +i \pm a$. sin. A' $\pm b$. sin. B'.
(4) $\Sigma l = +a$. cos. A' $\pm b$. cos. B'.
(5) Tan. C = $\Sigma d \div \Sigma l$.
(18) Cos. T = $(a \pm b)$. sin. C $\div \Sigma d = (a \pm b)$. cos. C $\div \Sigma l$.
(19) $t = \Sigma d$. sin. T $\div$ sin. C = Σl . sin. T $\div$ cos. C = $(a \pm b)$. tan. T.

Prop. IV. CN . CTa = CN . CZ ± CZ . CTa = (C ± T). (20)

(1) CN . CA = { +A' when the curve through A is convex.
−A' " " " " concave.

(2) B = CN . BZ = { B' when the centres are on the same side of **AB**.
180°−B' when the centres are on different sides of **AB**.

B is { positive when the curve through B is convex.
negative " " " " concave.

By proposition X., the tangential deflection over an arc from the extremity of one radius to the extremity of another is the same in sign and magnitude with the deflection from the initial to the terminal radial direction, when both directions are estimated either from centre to tangent or from tangent to centre.

The tangential deflection Ia, *from* A *to* Ta, is therefore equal to CA.CTa, and the tangential deflection Ib, from Tb to B, to TbZ.BZ.

Proposition V. $CA.CT_a = CN.CT_a - CN.CA = (C \pm T) - A.$

$T_bZ.BZ = CN.BZ - CN.T_bZ = B - CN.T_bZ.$

As CTa and Tbz are the same direction when the curves are reversed, as in figure 90, and opposite directions when, as in figures 91 and 92, they are compound, CN.TbZ is equal to CN.CA when the curves are reversed, and to CN.CA + 180° when they are compound, and as Ia and Ib are equal to CA.CTa and TbZ.BZ,

$$I_a = (C \pm T) - A. \quad (21).$$

$$I_b = \begin{cases} B-(C \pm T) & \text{when the curves are reversed.} \\ B-(C \pm T)+180° & \text{" " " compound.} \end{cases} \quad (22)$$

If the calculated value of Ia or Ib is greater than 180°, subtract from 360° and change the sign. In 22, give to 180° a sign opposite to that of the difference B−(C ± T), so that the value of Ib may be less than 180°. This is, however, only for convenience, as a plus or minus sign may be used at pleasure for 180°.

In (21) and (22), (C−T) will give one value of Ia or Ib, and (C+T) the other, the difference in the two values being, therefore, equal to 2 T.

The algebraic sum, Ia+Ib, should be equal to the angle of intersection, determined by the rules of problem V.

6. *Problem XXVI.*—In problem XXV., the radii, a and b, of the two curves were given, to determine the length of the common tangent. In this problem, t and one of the radii are given, to determine the other radius.

For this purpose assume, as in problem XXV., an axis, CN,

at right angles to **AB**, and determine the formulæ for Σd, Σl, and c as follows:

(3) $\Sigma d = i \pm a$. sin. A $\pm b$. sin. B.
(4) $\Sigma l = a$. cos. A $\pm b$. cos. B.
(7) and (16) $c^2 = (\Sigma d)^2 + (\Sigma l)^2 = t^2 + (a \pm b)^2$. (1)

The signs of the terms of these equations may be determined by the rules given in article 4, but, for the solution of the general problem, they are assumed to be plus, and changed in the application to particular cases.

In (1), squaring both members of the equation; substituting $(a^2 + b^2)$ for a^2 . (sin.^{2} A + cos.^{2} A) + b^2 . (sin.^{2} B + cos.^{2} B); cancelling $(a^2 + b^2)$ in each member; transposing t^2 to the first member, and dividing by $2i$, the following equation is obtained:

$$\left(\frac{i^2 - t^2}{2i}\right) + a \text{ . sin. A} + b \text{ . sin. B} + \frac{a.b}{i} (\text{cos. A . cos. B} + \text{sin. A . sin. B}) = \frac{a.b}{i}. \quad (2)$$

Prop. XII. (2). Cos. A . cos. B + sin. A . sin. B = cos. (A − B) = cos. R.

Substituting this value of (cos. A . cos. B + sin. A . sin. B) in (2),

$$\frac{(i^2 - t^2)}{2i} + a \text{ . sin. A} + b \text{ . sin. B} + \frac{a.b}{i} \text{ . cos. R} = \frac{a.b}{i}. \quad (3)$$

The term $a.b$, in the second member of this equation, is obtained by squaring $(a \pm b)$ in (1), and, as the arithmetic value of this term is $(a + b)$ for reversed curves, and $(a - b)$ for compound curves, the sign prefixed to $a \,.\, b \div i$, in the second member of the equation, will be plus when the curves are reversed, and minus when they are compound. The term $a.b.$ cos. R $\div i$, in the first member, is derived from the squares of Σd and Σl, and the sign will depend upon whether R is acute or obtuse.

Transposing this term, the second member will be:

$$\left.\begin{array}{l} +a.b.(1 - \text{cos. R}) \div i = +a.b. \text{ vers. R when the curves are reversed.} \\ -a.b.(1 + \text{cos. R}) \div i = -a.b. \text{ vers. } (180° - \text{R}) \text{ when the curves are compound.} \end{array}\right\} \quad (4)$$

The terms A and B in these equations are the bearings, CN.CA and CN.BZ, of the radii (**CA**) and (**BZ**), and, by proposition V.,

$$BZ.CA=CN.CA-CN.BZ=(A-B)=R.$$

If $I=Ia+Ib=(A'\pm B')$, or $180°-(A'\pm B')$, is the angle of intersection of the tangents through A and B,

Prop. X. (4). $I=\begin{cases} R \text{ when the curves are reversed.} \\ 180°-R \text{ when the curves are compound.} \end{cases}$

Substituting these values of R and $180°-R$ in (4) and (3),

$$(i^2-t^2)\div 2i+a\,.\,\text{sin. A}+b\,.\,\text{sin. B}=\pm a\,.\,b\,.\,\text{vers. I}\div i. \quad (5)$$

$$a=\frac{\frac{1}{2}(i^2-t^2)\div i+b.\text{sin. B}}{(-\text{sin. A})\pm b.\text{ vers. I}\div i} \qquad b=\frac{\frac{1}{2}(i^2-t^2)\div i+a.\text{sin. A}}{(-\text{sin. B})\pm a.\text{vers. I}\div i} \quad (6)$$

In (6a), the sign to be prefixed to $\frac{1}{2}(i^2-t^2)\div i$ will be plus, unless t is greater than i, in which case the term will be negative.

b. sin. B is the departure of the radius, (**BZ**), and is positive when the curve through B is convex, and negative when it is concave.

In the denominator, (−sin. A) was obtained by the transposition of the departure, a. sin. A, from the first to the second member of (5), and, as the sign was thereby changed, the term is positive when the curve through A is concave, and negative when it is convex.

For the reasons above stated, the second term of the denominator will be plus for reversed, and minus for compound, curves.

The signs in (6b) are determined in the same manner.

If $\frac{1}{2}(i^2-t^2)\div i+a$. sin. $A+b$. sin. B had been transposed to the second member of (3), and $a.b\div i$ to the first, the result would have been the same, except that the signs of ALL the terms would have been different. If, then, for any particular form of curves, the equation has been determined by the rules above, the solution will not be possible for any radius which, substituted in the second member, will give different signs to the numerator and denominator.

Formulæ (2), problem XXVI., is derived from (1), and the

signs of the corresponding terms in the two equations are determined by the same rules.

7. *Problem XX.* (figures 71 and 72).—Formula (6b) of the last article may be placed under the following form:

Problem XXVI. (2). $b=(o \pm a) \,.\, n \div (p \pm a)$.

In this equation, $o=\frac{1}{2}(i^2-t^2)\div i$.sin. A. $n=i$.sin. A$\div$vers. I. $p=i$.sin. B$\div$vers. I.

If o and a have the same magnitudes and different signs, the numerator, $(o-a)$, and the radius, b, reduce to zero; the circle through B reduces to a point, which is at the extremity, B, of the tangent (**TB**)$=t$, and (**CB**)$=c$ is the line from one centre to the other.

The conditions making b equal to zero are independent of the value of the angle B, and as $(a=o)$, $(b=$ zero), and t are known, the tangential deflection, Ia, from A to Tb, may be determined by the formulæ, for problem XXV., given in article 5 of this proposition, as follows:

(3) $\Sigma d = +i \pm a$. sin. A$' \pm b$. sin. B$' = i \pm a$.sin. A$'$.
(4) $\Sigma l = +a$.cos. A$' \pm b$.cos. B$' = a$.cos. A$'$.
(5) Tan. C $= \Sigma d \div \Sigma l$.
(14) Tan. T $= t \div a$.
(1) A$=\begin{cases} +\text{A}' \text{ when the curve through A is convex} \\ -\text{A}' \quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“ concave} \end{cases}$ to **AB**.
(21) Ia = (C $\pm$ T)$-$A.
(21) Ib = B$-$(C $\pm$ T) = 0, and therefore B$=$(C $\pm$ T).

The formulæ for Ia, in problem XX., require somewhat less calculation than the above, and are derived from the triangle ABTb, in which ABTb=B; TbAB=A$\pm\frac{1}{2}$ Ia; ATbB$=\frac{1}{2}$ Ia; (**AB**)$=i$, and (**BT**b)$=t$.

When a is given, to determine t, $o=a=\frac{1}{2}(i^2-t^2)\div i$.sin. A, and therefore $t=\sqrt{i.(i \pm 2a \,.\, \text{sin. A}')}$.

8. *Problem XIX.* (figures 73 to 76).—If, in equation (2b) of the last article, p and a have equal magnitudes and different signs, the denominator will be zero, b will be infinite, and the curve through B will coincide with the tangent **BI**.

In this case the curve through A, with radius p, will touch the tangent **BI** at a point represented in the figures by Tb, and, as the conditions making b equal to zero are independent

of the value of t, it may be assumed equal to the distance (**T***b***B**) from T*b* to B.

As the extremity of this tangent is at the point B, the corresponding value of b must be zero, and a must therefore be equal to o in order that the numerator may be zero; but a is also equal to p, and therefore the value of t may be obtained by equating o and p, in the following manner:

$o = p = \frac{1}{2}(i^2 - t^2) \div i$. sin. $A' = i$. sin. B $\div$ vers. I.

$$t^2 = \frac{i^2 \text{ . (vers. I} - 2 \text{ . sin. A . sin. B)}}{\text{vers. I}} \text{ when } t \text{ is less than } i. \quad \text{(a)}$$

$$t^2 = \frac{i^2 \text{ . (vers. I} + 2 \text{ . sin. A . sin. B)}}{\text{vers. I}} \text{ when } t \text{ is greater than } i. \quad \text{(b)}$$

It may be proven that t is less than i when, as in figure 71 or 76, I is equal to (A′+B′) or to 180°−(A′−B′), and greater than i when, as in figure 74 or 75, I is equal to (A′−B′) or to 180°−(A′+B′).

The value of t is, therefore, to be obtained from (a) when I = (A′+B′) or 180°−(A′−B′), and from (b) when I = (A′−B′) or 180°−(A′+B′).

Vers. [I = (A′+B′)] = 1 − cos. (A′+B′) = 1−cos. A . cos. B + sin. A . sin. B.

Vers. I − 2 . sin. A . sin. B = 1−cos. A . cos. B−sin. A . sin. B = 1−cos. (A′−B′) = vers. (A′−B′).

By similar reasoning it may be shown that—

Vers. [I=(A′+B′)]−2 . sin. A . sin. B=vers. (A′−B′).

Vers. [I = 180°− (A − B)] − 2 . sin. A . sin. B = vers. [180°− (A′+B′)].

Vers. [I=(A′−B′)]+2 . sin. A . sin. B=vers. (A′+B′).

Vers. [I = 180°− (A′+B′)]+2 . sin. A . sin. B = vers. [180°− (A′−B′)].

Substituting these values in (a) and (b),

$$\text{When I} = (A'+B'),\ t^2 = \frac{i^2 \text{ . vers. } (A'-B')}{\text{vers. } (A'+B')} = \frac{i^2 \text{ . sin.}^2 \frac{1}{2}(A'-B')}{\text{sin.}^2 \frac{1}{2}(A'+B')}$$

$$\text{When I} = (A-B),\ t^2 = \frac{i^2 \text{ . vers. } (A'+B')}{\text{vers. } (A'-B')} = \frac{i^2 \text{ . sin.}^2 \frac{1}{2}(A'+B')}{\text{sin.}^2 \frac{1}{2}(A'-B')}$$

$$\text{When } I=180°-(A'+B'),\ t^2=\frac{i^2.\text{vers.}\,[180°-(A'-B')]}{\text{vers.}\,[180°-(A'+B')]}=\frac{i^2.\cos.^2\tfrac{1}{2}(A'-B')}{\cos.^2\tfrac{1}{2}(A'+B')}$$

$$\text{When } I=180°-(A-B),\ t^2=\frac{i^2.\text{vers.}\,[180°-(A'+B')]}{\text{vers.}\,[180°-(A'-B')]}=\frac{i^2.\cos.^2\tfrac{1}{2}(A'+B')}{\cos.^2\tfrac{1}{2}(A'-B')}$$

When, therefore, b is infinite, the distance, t, from B to the point, Tb, of contact with the curve through A, on the tangent **BI**, will be as follows:

$$\left.\begin{array}{l}\text{When } I=(A'+B'),\ t=\dfrac{i.\sin.\tfrac{1}{2}(A'-B')}{\sin.\tfrac{1}{2}(A'+B')}=\dfrac{i.(\cos.B'-\cos.A')}{\text{vers. I}}\\[2ex]\text{When } I=(A'-B'),\ t=\dfrac{i.\sin.\tfrac{1}{2}(A'+B')}{\sin.\tfrac{1}{2}(A'-B')}=\dfrac{i.(\cos.B'-\cos.A')}{\text{vers. I}}\\[2ex]\text{When } I=180°-(A'+B'),\ t=\dfrac{i.\cos.\tfrac{1}{2}(A'-B')}{\cos.\tfrac{1}{2}(A'+B')}=\dfrac{i.(\cos.B'+\cos.A')}{\text{vers. I}}\\[2ex]\text{When } I=180°-(A'-B'),\ t=\dfrac{i.\cos.\tfrac{1}{2}(A'+B')}{\cos.\tfrac{1}{2}(A'-B')}=\dfrac{i.(\cos.B'+\cos.A')}{\text{vers. I}}\end{array}\right\}(1)$$

When a is infinite, the distance, (**AT**a)=t, from A to the point, Ta, at which the curve through B touches the tangent **AI**, will be the same as the distance (**BT**b) = t when b is infinite, the two tangents being equal, as in all other cases of problems XXV. and XXVI.

9. *Problem XXIII.*—In this problem it is required to find equal radii for a given value of t in the case of reversed curves.

To obtain the formula for any particular figure, determine the equation between a and b by the rules of problem XXVI. If, then, a is made equal to b, a quadratic equation will result, from which the value of the equal radii may be deduced. In this manner the equations of table XI. have been determined.

10. *Problem XXVII.*—In this problem, one of the radii, as a, is given, and also the tangential deflection, Ia, from A to the terminal point, Ta, of the curve, and it is required to determine the other radius, b, the point, Tb, at which the curve will touch the tangent produced from Ta, and the tangential deflection, Ib, from Tb to B. I, equal to Ia+Ib, is first to be determined by the rules of problem V., and from this equation, when one of the deflections, Ia or Ib, is given, the other may be determined as shown in the table of problem XXVII.

The bearing, (C ± T), of the radius from C to the extremity, Ta, of the tangent, (**T**a**T**b)=t, is then to be determined in the following manner:

(1) Art. (3). A = { +A′ when the curve through A is convex to **AB**. / −A′ when the curve through A is concave to **AB**. }

(2) B = { B′ for centres on different sides / 180° − B′ for centres on the same side } of **AB**.

B is positive when the curve through B is convex to **AB**.
" negative " " " concave "

(21) (C ± T)=Ia + A.

(22) (C ± T) = B − Ib when the curves are reversed; =B − Ib + 180° when they are compound.

Ia, Ib, and (C ± T) being thus determined, the equations for t, and a or b, may be obtained in the following manner:

In any figure, as, for instance, 101, assume the radius **CT**a as the co-ordinate axis, and C as the origin, and determine the bearings, departures, and latitudes of the lines, (**CA**)=a; (**AB**)=i; (**BZ**)=b; (**ZT**b)=b; (**T**b**T**a)=t; and (**T**a**C**)=a.

Since Ia is the deflection from the axis CTa to CA, the bearing of CA is Ia degrees to the left, or −Ia, and the bearing of any other line is equal to the algebraic sum of the deflection to that line from the preceding line and the bearing of that line.

CTa.CA=−Ia; CTa.AB=−Ia+90°−A′=+90°−(Ia−A) = + 90° − (C ± T); CTa . BZ = − Ib; CTa . ZTb = − 180°; CTa . TbTa = − 180° + 90° = −90°; CTa . TaC = 180°. With these bearings, the distances, departures, and latitudes will be as follows:

Distances.	Bearings.	Departures.	Latitudes.
(**CA**) = a	−Ia	−a . sin. Ia	+a . cos. Ia
(**AB**) = i	+90°−(C ± T)	+i . cos. (C ± T)	+i . sin. (C ± T)
(**BZ**) = b	− Ib	−b . sin. Ib	+b . cos. Ib
(**ZT**b) = b	180°	0	−b
(**T**b**T**a)=t	−90°	−t	0
(**T**a**C**) = a	180°	0	−a

As these lines form a closed polygon, the algebraic sums of the departures and latitudes must be equal to zero, and, therefore,

$\Sigma d = -a$. sin. I$a + i$. cos. (C $\pm$ T)$-b$. sin. I$b - t = 0$.

$\therefore t = +i$. cos. (C $\pm$ T)$-a$. sin. I$a - b$. sin. Ib.

$\Sigma l = +a$. cos. I$a + i$. sin. (C $\pm$ T)$+b$. cos. I$b - b - a = 0$.

$\therefore i$. sin. (C $\pm$ T)$= a$. (1$-$cos. Ia)$+b$. (1$-$cos. Ib)$= a$. vers. I$a +$ b . vers. Ib.

These are the same with the equations of the table in problem XXVII. for figure 104. The equations may also be obtained by construction of the figure in the following manner:

From A and B, figure 116¼, let fall the perpendiculars, **AP***a* and **BP***b*, on the parallel tangents **CT***a* and **ZT***b*, and from B a perpendicular **BR** on **AP***a*. (**BR**) $= i$. sin. (C $\pm$ T) is the perpendicular distance from (**BP***b*)$= b$. sin. Ia to (**AP***a*) $= a$. sin. Ia, and is equal to the difference (b . vers. I$b - a$. vers. Ia), in accordance with formula (3), problem XXVII.

(**AR**)$= i$. cos. (C $\pm$ T) is parallel to (**T***a***T***b*)$= t$, and t is equal to b . sin. I$b - a$. sin. I$a +$ (**AR**), in accordance with formula (4).

11. *Problem XVI., formula* (13).—In problem XXVI., t and one of the radii are given, to determine the other radius. In problem XVI., the curves are produced to intersection at an angle F, and F and one of the radii are given, to determine the other radius.

The formula is obtained from the formula of problem XXVI. by making t equal to zero and substituting (vers. I$-$vers. F) for vers. I in the denominator.

In figure 53, C and Z are the centres of the intersecting curves; a and b their radii; (**T***a***T***b*)$= t$ their common tangent, and F the deflection between the tangents intersecting at F.

In the triangle CFZ the sides are: (**CF**)$= b$; (**FZ**)$= a$; and (**ZC**)$= c$, and the angle CFZ at the vertex is equal to F.

Assuming (**CZ**)$= c$ as the closing line to (**CF**) and (**FZ**), with reference to CF as the axis, the bearings, distances, departures, and latitudes will be as follows:

Distances.	Bearings.	Departures.	Latitudes.
(**CF**) $= b$	CF . CF $= 0$	b . sin. $0 = 0$	b . cos. $0 = +b$
(**FZ**) $= a$	CF . FZ $= 180° -$ F	$+a$. sin. F	$-a$. cos. F

$$\Sigma d = + a \,.\, \text{sin. F.} \qquad \Sigma l = b \,.\, - a \,.\, \text{cos. F.}$$
$$c^2 = (\Sigma d)^2 + (\Sigma l)^2 = a^2 + b^2 - 2 \,.\, a \,.\, b \,.\, \text{cos. F.}$$

As the curves intersect, they are compound with reference to their common tangent, and, by (16), article 5,

$$c^2 = (a - b)^2 + t^2.$$

Equating these two values of c,

$$(a-b)^2 + t^2 = a^2 + b^2 - 2 \,.\, a \,.\, b \,.\, \text{cos. F.}$$
$$\therefore \quad \tfrac{1}{2} t^2 = a \,.\, b \,.\, (1 - \text{cos. F}) = a \,.\, b \,.\, \text{vers. F.}$$

Transferring from the first to the second member of (3), article (6), the term $\frac{1}{2} t^2 \div i$,

$$\tfrac{1}{2} i + a \,.\, \text{sin. A} + b \,.\, \text{sin. B} = \tfrac{1}{2} \frac{t^2}{i} - \frac{a \,.\, b \,.\, \text{vers. I}}{i}.$$

Substituting in this equation the value of $\frac{1}{2} t^2$ above,

$$\tfrac{1}{2} i + a \,.\, \text{sin. A} + b \,.\, \text{sin. B} = - \frac{a \,.\, b}{i} \,.\, (\text{vers. I} - \text{vers. F}).$$

The formulæ for a and b, deduced from this equation, are in all respects the same with the formulæ of problem XXVI. for compound curves with t equal to zero, except that (vers. I—vers. F) is substituted in the denominator for vers. I; the signs of the terms are determined by the same rules.

The deflections, Xa and Xb, over the crossing curves may be determined by methods analogous to those of problem XXV. for determining Ia and Ib, but they may be determined with less calculation by the formulæ of table III., problem XVI.

TABLE I., PROBLEM XIII. (14)

GIVING THE ORDINATES (e) FROM TANGENT TO CURVE AT A DISTANCE (n) FROM TANGENT-POINT.

Degree of Curve.	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	Chord 100′.	
																					e	n
1°	.00	.01	.02	.03	.05	.08	.11	.14	.18	.22	.26	.31	.37	.43	.49	.56	.63	.71	.79	.87	.87	100
2°	.00	.02	.04	.07	.11	.16	.21	.28	.35	.44	.53	.63	.74	.86	.98	1.12	1.26	1.41	1.58	1.75	1.75	99.98
3°	.01	.03	.06	.11	.16	.24	.32	.42	.53	.65	.79	.94	1.11	1.29	1.47	1.68	1.89	2.12	2.37	2.62	2.62	99.97
4°	.01	.04	.08	.14	.22	.31	.43	.56	.71	.87	1.06	1.26	1.48	1.72	1.96	2.24	2.52	2.83	3.16	3.49	3.49	99.94
5°	.01	.04	.10	.18	.27	.39	.53	.70	.88	1.09	1.32	1.57	1.85	2.15	2.45	2.80	3.15	3.53	3.94	4.36	4.36	99.90
6°	.01	.05	.12	.22	.33	.47	.64	.84	1.06	1.31	1.58	1.89	2.21	2.57	2.95	3.36	3.78	4.24	4.73	5.24	5.23	99.86
7°	.02	.06	.14	.25	.38	.55	.75	.98	1.24	1.53	1.85	2.20	2.58	3.00	3.44	3.92	4.42	4.96	5.53	6.12	6.10	99.81
8°	.02	.07	.16	.28	.44	.63	.86	1.12	1.42	1.75	2.11	2.51	2.95	3.43	3.93	4.48	5.05	5.67	6.33	7.01	6.98	99.76
9°	.02	.08	.18	.31	.49	.71	.96	1.26	1.59	1.97	2.38	2.83	3.32	3.86	4.43	5.04	5.69	6.39	7.13	7.90	7.85	99.69
10°	.02	.09	.20	.35	.55	.79	1.07	1.40	1.77	2.18	2.64	3.14	3.69	4.29	4.92	5.61	6.33	7.11	7.92	8.78	8.72	99.62
11°	.02	.10	.22	.39	.60	.86	1.18	1.54	1.95	2.40	2.90	3.45	4.06	4.72	5.42	6.17	6.97	7.82	8.73	9.67	9.58	99.54
12°	.03	.11	.23	.42	.65	.94	1.28	1.68	2.12	2.62	3.17	3.77	4.43	5.15	5.92	6.74	7.61	8.54	9.53	10.57	10.45	99.45
13°	.03	.12	.25	.45	.70	1.02	1.39	1.82	2.30	2.84	3.43	4.08	4.80	5.58	6.41	7.30	8.25	9.27	10.34	11.47	11.32	99.36
14°	.03	.12	.27	.48	.76	1.10	1.50	1.96	2.48	3.06	3.69	4.40	5.17	6.01	6.91	7.87	8.90	9.99	11.15	12.37	12.19	99.25
15°	.03	.13	.30	.52	.82	1.18	1.61	2.10	2.66	3.28	3.96	4.72	5.55	6.45	7.41	8.44	9.55	10.72	11.97	13.28	13.05	99.14

TABLE II. (A, B, C), PROBLEM XV.

COMPOUND ARCS.

	Station	Alignment	Short Chords	Deflection to Long Chord $K\sigma$	Long Chord K	Deflection from Long Chord $K\zeta$	Total Deflection $I = K\sigma + K\zeta$	Offsets from Tangent	
								n	o
Table A — Curvature increases 1° 00 per ¼ Station.	.25	E.C.C. 1°	25.00	0° 7′ 30″	25.00	0° 7′ 30″	0° 15′	25.00	0.05
	.50	E.C.C. 2°	25.00	0° 18′ 40″	50.00	0° 26′ 20″	0° 45′	50.00	0.27
	.75	E.C.C. 3°	25.00	0° 35′ 00″	75.00	0° 55′ 00″	1° 30′	74.99	0.76
	1.00	E.C.C. 4°	25.00	0° 56′ 15″	100.00	1° 33′ 45″	2° 30′	99.98	1.64
	.25	E.C.C. 5°	25.01	1° 22′ 30″	124.99	2° 22′ 30″	3° 45′	124.95	3.00
	.50	E.C.C. 6°	25.01	1° 53′ 45″	149.97	3° 21′ 15″	5° 15′	149.88	4.96
	.75	E.C.C. 7°	25.01	2° 30′ 00″	174.92	4° 30′ 00″	7° 00′	174.76	7.63
	2.00	E.C.C. 8°	25.02	3° 11′ 15″	199.54	5° 48′ 45″	9° 00′	199.53	11.11
	.25	E.C.C. 9°	25.02	3° 57′ 30″	224.70	7° 17′ 30″	11° 15′	224.17	15.51
	.50	E.C.C. 10°	25.03	4° 48′ 50″	249.49	8° 56′ 10″	13° 45′	248.60	20.93
	.75	E.C.C. 11°	25.04	5° 45′ 10″	274.15	10° 44′ 50″	16° 30′	272.77	27.46
	3.00	E.C.C. 12°	25.04	6° 46′ 20″	298.67	12° 43′ 40″	19° 30′	296.59	35.22
Table B — Curvature increases 2° 00 per ¼ Station.	.25	E.C.C. 2°	25.00	0° 15′ 00″	25.00	0° 15′ 00″	0° 30′	25.00	0.11
	.50	E.C.C. 4°	25.00	0° 37′ 30″	50.00	0° 52′ 30″	1° 30′	50.00	0.54
	.75	E.C.C. 6°	25.01	1° 10′ 00″	75.00	1° 50′ 00″	3° 00′	74.99	1.53
	1.00	E.C.C. 8°	25.02	1° 52′ 30″	100.00	3° 07′ 30″	5° 00′	99.95	3.27
	.25	E.C.C. 10°	25.03	2° 45′ 00″	124.96	4° 45′ 00″	7° 30′	124.93	6.00
	.50	E.C.C. 12°	25.04	3° 47′ 30″	149.88	6° 42′ 30″	10° 30′	149.85	9.91
Table C — Curvature increases 3° 00 per ¼ Station.	.25	E.C.C. 3°	25.00	0° 22′ 30″	25.00	0° 22′ 30″	0° 45′	25.00	0.16
	.50	E.C.C. 6°	25.01	0° 56′ 15″	50.00	1° 48′ 45″	2° 15′	49.99	0.82
	.75	E.C.C. 9°	25.02	1° 45′ 00″	75.00	2° 45′ 00″	4° 30′	74.97	2.29
	1.00	E.C.C. 12°	25.04	2° 48′ 00″	99.98	4° 42′ 00″	7° 30′	99.98	4.91

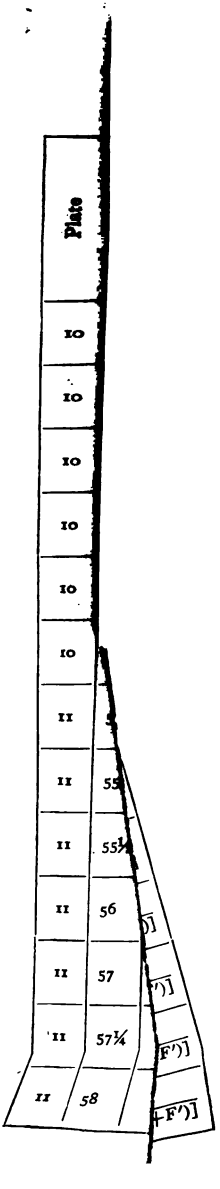
Plate
10
10
10
10
10
10
11
11
55
11
55¼
11
56
11
57
11
57¼
11
58

TABLE IV., PROBLEM XVIII.

TURN-OUTS FROM SWITCHES. GAUGE, ($g = 4.708$); THROW, ($s = .417$).

Frog-angle.		Frog-distances.			Dø.		Frog-angle.		Frog-distances.			Dø.		Ds = Dø − C.		
F+S	Diff. 1′ of Dø.	(AF) k_a	Diff. 1′ of F+S	(BF) k_b	S = σ	Diff. 1′ of F	F+S	Diff. 1′ of Dø	(AF) k_a	Diff. 1′ of F+S	(BF) k_b	S = σ	Diff. 1′ of F	S	C	Diff. 1′ of S
5° 00′	0′.5	98.3	.31′	98.4	5° 05′	2′.1	9° 30′	0′.3	51.6	.09′	51.8	18° 24′	3′.9	40′	0° 06′	
10′	0′.5	95.1	.30′	95.2	5° 26′	2′.1	40′	0′.3	50.7	.08′	50.9	19° 03′	3′.0	50′	0° 09′	.3′
20′	0′.5	92.1	.27′	92.2	5° 47′	2′.1	50′	0′.2	49.9	.08′	50.1	19° 43′	4′.0	1° 00′	0° 12½′	.3′
30′	0′.5	89.3	.26′	89.4	6° 09′	2′.2	10° 00′	0′.2	49.0	.08′	49.2	20° 24′	4′.1	10′	0° 17′	.4′
40′	0′.5	86.7	.25′	86.8	6° 32′	2′.3	10′	0′.2	48.2	.08′	48.4	21° 05′	4′.1	20′	0° 22′	.5′
50′	0′.5	84.2	.23′	84.3	6° 55′	2′.3	20′	0′.2	47.4	.07′	47.6	21° 47′	4′.2	30′	0° 28′	.6′
6° 00′	0′.4	81.9	.23′	82.0	7° 19′	2′.4	30′	0′.2	46.7	.07′	46.9	22° 30′	4′.3	40′	0° 35′	.7′
10′	0′.4	79.7	.21′	79.8	7° 44′	2′.5	40′	0′.2	46.0	.07′	46.2	23° 14′	4′.4	50′	0° 42′	.7′
20′	0′.4	77.6	.20′	77.7	8° 09′	2′.5	50′	0′.2	45.3	.07′	45.5	23° 58′	4′.4	2° 00′	0° 50′	.8′
30′	0′.4	75.6	.18′	75.7	8° 35′	2′.6	11° 00′	0′.2	44.6	.07′	44.8	24° 43′	4′.5	10′	0° 59′	.9′
40′	0′.4	73.7	.18′	73.8	9° 02′	2′.7	10′	0′.2	43.9	.06′	44.1	25° 29′	4′.6	20′	1° 08′	.9′
50′	0′.3	71.9	.18′	72.0	9° 30′	2′.8	20′	0′.2	43.3	.06′	43.5	26° 16′	4′.7	30′	1° 18′	1′.
7° 00′	0′.3	70.2	.16′	70.3	9° 58′	2′.8	30′	0′.2	42.6	.06′	42.8	27° 03′	4′.7	40′	1° 29′	1′.1
10′	0′.3	68.5	.16′	68.6	10° 27′	2′.9	40′	0′.2	42.0	.06′	42.2	27° 51′	4′.8	50′	1° 41′	1′.2
20′	0′.3	66.9	.15′	67.1	10° 56′	2′.9	50′	0′.2	41.4	.06′	41.6	28° 40′	4′.9	3° 00′	1° 53′	1′.2
30′	0′.3	65.4	.14′	65.6	11° 26′	3′.0	12° 00′	0′.2	40.8	.05′	41.0	29° 30′	5′.0	10′	2° 06′	1′.3
40′	0′.3	64.0	.13′	64.2	11° 57′	3′.1	10′		40.3	.05′	40.5			20′	2° 19′	1′.3
50′	0′.3	62.7	.13′	62.8	12° 29′	3′.2	20′		39.8	.05′	40.0			30′	2° 33′	1′.4
8° 00′	0′.3	61.4	.13′	61.5	13° 02′	3′.3	30′		39.2	.05′	39.4			40′	2° 48′	1′.5
10′	0′.3	60.1	.12′	60.3	13° 35′	3′.3	40′		38.7	.05′	38.9			50′	3° 03′	1′.5
20′	0′.3	58.9	.12′	59.1	14° 08′	3′.3	50′		38.2	.05′	38.4			4° 00′	3° 20′	1′.7
30′	0′.3	57.7	.12′	57.9	14° 43′	3′.5	13° 00′		37.7	.05′	37.9			10′	3° 37′	1′.7
40′	0′.3	56.6	.11′	56.8	15° 18′	3′.5	10′		37.2	.05′	37.4					
50′	0′.3	55.5	.11′	55.7	15° 54′	3′.6	20′		36.8	.05′	37.0					
9° 00′	0′.3	54.5	.10′	54.7	16° 30′	3′.6	30′		36.3	.05′	36.5					
10′	0′.3	53.5	.10′	53.7	17° 07′	3′.7	40′		35.9	.04′	36.1					
20′	0′.3	52.5	.09′	52.7	17° 45′	3′.8	50′		35.4	.04′	35.6					
							14° 00′		35.0	.04′	35.2					

TABLE V., PROBLEM XVIII.

TURN-OUTS FROM SWITCHES.

THREE-THROW SWITCH. GAUGE, $(g) = 4.708$; THROW, $(s) = 0.417$. $(g - 2s) = 3.874$.

Middle Frog-angle.		Frog-distances in feet.			Degree D″$_o$		Middle Frog-angle.		Frog-distances in feet.			Degree D″$_o$		D″$_s$ = D″$_o$ − C″.		
F″ + 2S	Diff. for 1′ of D″$_o$	(AF)″ k''_a	Diff. for 1′ of F″ + 2S	(BF)″ k''_b	S = o	Diff. for 1′ of F″	F″ + 2S	Diff. for 1′ of D″$_o$	(AF)″ k''_a	Diff. for 1′ of F″ + 2S	(BF)″ k''_b	S = o	Correction for 1′ of F″	S	C″	Diff. 1′ of S
7° 00′	0′.1	63.3	.14′	63.5	11° 02′	6′.5	12° 00′	0′.1	36.9	.05	37.1	32° 46′	11′.4	40′	0° 24′	1′.4
20′	0′.1	60.5	.13′	60.7	12° 07′	6′.8	20′	0′.1	35.9	.05′	36.1	34° 40′	11′.6	50′	0° 38′	1′.7
40′	0′.1	57.8	.12′	58.0	13° 15′	7′.1	40′	0′.1	34.9	.05′	35.1	36° 36′	12′.2	1° 00′	0° 55′	1′.9
8° 00′	0′.1	55.4	.11′	55.6	14° 26′	7′.4	13° 00′	0′.1	34.0	.04′	34.2	38° 38′		10′	1° 14′	2′.2
20′	0′.1	53.2	.11′	53.4	15° 40′	7′.7	20′		33.2	.04′	33.5			20′	1° 36′	2′.5
40′	0′.1	51.1	.10′	51.3	16° 57′	8′.0	40′		32.4	.04′	32.6			30′	2° 03′	2′.7
9° 00′	0′.1	49.2	.09	49.4	18° 17′	8′.3	14° 00′		31.6	.04′	31.8			40′	2° 30′	3′.0
20′	0′.1	47.5	.08′	47.7	19° 40′	8′.6	20′		30.8	.04′	31.0			50′	3° 02′	3′.4
40′	0′.1	45.8	.07′	46.0	21° 06′	9′.0	40′		30.1	.04′	30.3			2° 00′	3° 37′	3′.6
10° 00′	0′.1	44.3	.07′	44.5	22° 36′	9′.4	15° 00′		29.4	.03′	29.7			10′	4° 15′	3′.8
20′	0′.1	42.8	.07′	43.0	24° 10′	9′.6	20′		28 8	.03′	29.1			20′	4° 55′	4′.2
40′	0′.1	41.5	.06′	41.7	25° 46′	10′.0	40′		28.2	.03′	28.5			30′	5° 39′	4′.8
11° 00′	0′.1	40.2	.06′	40.4	27° 26′	10′.3	16° 00′		27.6		27.9			40′	6° 25′	5′.2
20′	0′.1	39.0	.06′	39.2	29° 09′	10′.7								50′	7° 56′	5′.4
40′	0′.1	37.9	.05′	38.1	30° 56′	11′.0								3° 00′	8° 10′	

TABLE VI., PROBLEM XVIII.

TURN-OUTS FROM SWITCHES. GAUGE, ($g = 3.00$); THROW, ($s = 0.333$).

Frog-angle.		Frog-distances.			Dσ.		Frog-angle.		Frog-distances.			Dσ.		Ds = Dσ − C.		
F	Diff. 1′ of Dσ	(AF)	Diff. 1′ of F+S	(BF)	S = σ	Diff. 1′ of F	F+S	Diff. 1′ of Dσ	(AF)	Diff. 1′ of F+S	(BF)	S = σ	Diff. 1′ of F	S	C	Diff. 1′ of S
5° 00′	0′.3	61.1	0′.20	61.1	8° 10′	3′.4	9° 30′	0′.2	32.1	0′.06	32.2	29° 48′	6′.5	40′	0° 10′	0′.4
10′	0′.3	59.1	0′.19	59.2	8° 44′	3′.5	40′	0′.2	31.6	0′.06	31.7	30° 53′	6′.6	50′	0° 14′	0′.6
20′	0′.3	57.2	0′.17	57.3	9° 19′	3′.5	50′	0′.2	31.0	0′.05	31.1	31° 59′	6′.7	1° 00′	0° 20′	0′.7
30′	0′.3	55.5	0′.16	55.6	9° 54′	3′.7	10° 00′	0′.2	30.5	0′.05	30.6	33° 06′	6′.8	10′	0° 27′	0′.8
40′	0′.3	53.9	0′.16	54.0	10° 31′	3′.8	10′	0′.1	30.0	0′.05	30.1	34° 14′	7′.0	20′	0° 35′	0′.9
50′	0′.3	52.3	0′.16	52.4	11° 09′	3′.9	20′	0′.1	29.5	0′.05	29.6	35° 24′	7′.2	30′	0° 44′	1′.1
6° 00′	0′.3	50.9	0′.15	51.0	11° 48′	4′.0	30′	0′.1	29.0	0′.04	29.1	36° 36′	7′.2	40′	0° 54′	1′.2
10′	0′.2	49.5	0′.14	49.6	12° 28′	4′.0	40′	0′.1	28.6	0′.04	28.7	37° 48′	7′.4	50′	1° 06′	1′.3
20′	0′.2	48.2	0′.13	48.3	13° 08′	4′.2	50′	0′.1	28.1	0′.04	28.2	39° 02′	7′.6	2° 00′	1° 18′	1′.4
30′	0′.2	47.0	0′.12	47.1	13° 50′	4′.3	11° 00′	0′.1	27.7	0′.04	27.8	40° 18′	7′.6	10′	1° 32′	1′.5
40′	0′.2	45.8	0′.12	45.9	14° 34′	4′.4	10′	0′.1	27.3	0′.04	27.4	41° 34′	8′.0	20′	1° 46′	1′.6
50′	0′.2	44.7	0′.11	44.8	15° 18′	4′.6	20′	0′.1	26.9	0′.04	26.9	42° 54′		30′	2° 02′	1′.7
7° 00′	0′.2	43.6	0′.10	43.7	16° 04′	4′.7	30′		26.5	0′.04	26.6			40′	2° 19′	1′.8
10′	0′.2	42.6	0′.10	42.7	16° 51′	4′.8	40′		26.1	0′.04	26.2			50′	2° 38′	1′.9
20′	0′.2	41.6	0′.09	41.7	17° 39′	4′.9	50′		25.7	0′.04	25.9			3° 00′	2° 57′	2′.0
30′	0′.2	40.7	0′.09	40.8	18° 28′	5′.0	12° 00′		25.3	0′.04	25.5			10′	3° 16′	2′.1
40′	0′.2	39.8	0′.08	39.9	19° 18′	5′.1	10′		25.0	0′.03	25.2			20′	3° 37′	2′.2
50′	0′.2	39.0	0′.08	39.1	20° 09′	5′.3	20′		24.7	0′.03	24.9			30′	3° 59′	2′.3
8° 00′	0′.2	38.1	0′.07	38.2	21° 02′	5′.4	30′		24.3	0′.03	24.5			40′	4° 22′	2′.5
10′	0′.2	37.4	0′.07	37.5	21° 56′	5′.5	40′		24.0	0′.03	24.2			50′	4° 47′	2′.7
20′	0′.2	36.6	0′.07	36.7	22° 51′	5′.6	50′		23.7	0′.03	23.9			4° 00′	5° 14′	2′.8
30′	0′.2	35.9	0′.07	36.0	23° 47′	5′.7	13° 00′		23.4	0′.03	23.6			10′	5° 42′	
40′	0′.2	35.2	0′.07	35.3	24° 44′	5′.8	10′		23.1	0′.03	23.3					
50′	0′.2	34.5	0′.06	34.6	25° 42′	5′.9	20′		22.8	0′.03	23.0					
9° 00′	0′.2	33.9	0′.06	34.0	26° 41′	6′.1	30′		22.5	0′.03	22.7					
10′	0′.2	33.3	0′.06	33.4	27° 42′	6′.2	40′		22.2	0′.03	22.4					
20′	0′.2	32.7	0′.06	32.8	28° 44′	6′.4	50′		21.9	0′.03	22.1					
							14° 00′		21.7	0′.02	21.9					

TABLE VII., PROBLEM XVIII.

TURN-OUTS FROM SWITCHES.

THREE-THROW SWITCHES. GAUGE, ($g = 3.00$); THROW, ($s = 0.33$). ($g - 2s = 2.333$.

Middle Frog-angle.		Frog-distances in feet.			Degree D″$_\theta$		Middle Frog-angle.		Frog-distances in feet.			Degree D″$_\theta$		D″$_x$ = D″$_\theta$ − C″.		
F″ + 2S	Correction for 1′ of 1″$_\theta$	(AF)″	Correction for 1′ of F″ + 2S	(BF)″	S = θ	Correction for 1′ of F″	F″ + 2S	Correction for 1′ of 1″$_\theta$	(AF)″	Correction for 1′ of F″ + 2S	(BF)″	S = θ	Correction for 1′ of F″	S	C″	Correction for 1′ of S
7° 00′	0′.10	38.1	.08′	38.2	18° 23′	10′.9	12° 00′		22.2	.05′	22.4			0° 40′	0° 40′	2′.4
20′	0′.10	36.4	.08′	36.5	20° 12′	11′.3	20′		21.6	.03′	21.8			1° 00′	1° 02′	2′.8
40′	0′.09	34.8	.08′	34.9	22° 05′	11′.9	40′		21.1	.03′	21.3			10′	1° 30′	3′.3
8° 00′	0′.08	33.4	.07′	33.5	24° 04′	12′.5	13° 00′		20.5	.03′	20.7			20′	2° 03′	3′.7
20′	0′.07	32.0	.06′	32.1	26° 09′	13′.1	20′		20.0	.03′	20.2			30′	2° 40′	4′.2
40′	0′.07	30.8	.06′	31.0	28° 20′	13′.6	40′		19.5	.03′	19.7			40′	3° 22′	4′.8
9° 00′	0′.07	29.6	.05′	29.8	30° 36′	14′.2	14° 00′		19.0	.02′	19.2			50′	4° 10′	5′.4
20′	0′.07	28.6	.05′	28.8	32° 58′	14′.8	20′		18.6	.02′	18.8			2° 00′	5° 03′	5′.7
40′	0′.07	27.6	.05′	27.8	35° 26′	15′.4	40′		18.2	.02′	18.4			10′	6° 00′	6′.2
10° 00′	0′.06	26.7	.05′	26.9	38° 02′	16′.0	15° 00′		17.8	.01′	18.0			20′	7° 02′	6′.8
20′	0′.06	25.8	.05′	26.0	40° 40′	16′.8	20′		17.4	.01′	17.6			30′	8° 10′	7′.2
40′	0′.06	25.0	.04′	25.2	43° 28′	17′.4	40′		17.0	.01′	17.2			40′	9° 22′	7′.8
11° 00′	0′.05	24.2	.04′	24.4	46° 22′	18′.2	16° 00′		16.6	.01′	16.8			50′	10° 40′	8′.2
20′	0′.05	23.5	.04′	23.7	49° 24′	19′.1								3° 00′	12° 02′	8′.8
40′		22.8	.03′	23.0	52° 34′										13° 30′	

TABLE VIII., PROBLEM XVIII.

SIDINGS. c=Distance between Parallel Track-Centres; g=Gauge; $(c-g)=i$=Distance between Rails at Point of Parallel Tangency. In this Table $(c-g)=10$ Feet.

$n=\frac{1}{2}$ cot. $\frac{1}{2}$F		$F=\frac{57.2}{n}$		Chords in feet.			$D\sigma=\frac{100\,F}{2in}$		$n=\frac{57.2}{F}$		$F=\frac{57.2}{n}$		Chords in feet.			$D\sigma=\frac{100\,F}{2in}$	
Frog-number	Diff. per 1′ of F	Frog-angle	Diff. per .01 of n in minutes	$(\mathbf{AF})=k_a=2in$	$(\mathbf{BF})=k_b=\frac{i}{\sin.\ \frac{1}{2}F}$	Diff. in k_a or k_b per .01 of n	Degree of turn-out, Straight main track	Diff. in $D\sigma$, in minutes, per 1′ of F	Frog-number	Diff. per 1′ of F	Frog-angle	Diff. per .01 of n in minutes	$(\mathbf{AF})=k_a=2in$	$(\mathbf{BF})=k_b=\frac{i}{\sin.\ \frac{1}{2}F}$	Diff. in k_a or k_b per .01 of n	Degree of turn-out, Straight main track	Diff. in $D\sigma$, in minutes, per 1′ of F
5.00	.008	11° 25′	1′.3	100	100.5	0.2	11° 22′	1′.9	9.00	.023	6° 22′	.4′	180	180.3	0.2	3° 32′	1′.2
5.25	.008	10° 53′	1′.2	105	105.5	0.2	10° 20′	1′.9	9.25	.023	6° 11′	.4′	185	185.3	0.2	3° 20′	1′.2
5.50	.009	10° 23′	1′.1	110	110.5	0.2	9° 24′	1′.8	9.50	.025	6° 01′	.4′	190	190.3	0.2	3° 09′	1′.1
5.75	.010	9° 56′	1′.0	115	115.4	0.2	8° 36′	1′.7	9.75	.028	5° 52′	.4′	195	195.3	0.2	3° 00′	1′.
6.00	.011	9° 32′	.9′	120	120.4	0.2	7° 55′	1′.6	10.00	.028	5° 43′	.4′	200	200.3	0.2	2° 52′	1′.
6.25	.012	9° 09′	.8′	125	125.4	0.2	7° 18′	1′.6	10.25	.030	5° 35′	.3′	205	205.3	0.2	2° 43′	1′.
6.50	.013	8° 48′	.8′	130	130.4	0.2	6° 45′	1′.5	10.50	.031	5° 27′	.3′	210	210.3	0.2	2° 35′	1′.
6.75	.014	8° 28′	.7′	135	135.4	0.2	6° 14′	1′.4	10.75	.033	5° 19′	.3′	215	215.2	0.2	2° 28′	1′.
7.00	.015	8° 10′	.7′	140	140.4	0.2	5° 40′	1′.3	11.00	.035	5° 12′	.3′	220	220.2	0.2	2° 21′	1′.
7.25	.016	7° 53′	.6′	145	145.3	0.2	5° 25′	1′.3	11.25	.036	5° 05′	.3′	225	225.2	0.2	2° 15′	.9′
7.50	.017	7° 38′	.6′	150	150.3	0.2	5° 05′	1′.3	11.50	.038	4° 59′	.2′	230	230.2	0.2	2° 10′	.9′
7.75	.018	7° 23′	.6′	155	155.3	0.2	4° 45′	1′.3	11.75	.040	4° 53′	.2′	235	235.2	0.2	2° 05′	.9′
8.00	.019	7° 09′	.5′	160	160.3	0.2	4° 27′	1′.2	12.00	.042	4° 47′	.2′	240	240.2	0.2	2° 00′	.9′
8.25	.020	6° 56′	.5′	165	165.3	0.2	4° 11′	1′.2	12.25	.042	4° 41′	.2′	245	245.2	0.2	1° 55′	.9′
8.50	.021	6° 44′	.4′	170	170.3	0.2	3° 56′	1′.2	12.50	.042	4° 35′	.2′	250	250.2	0.2	1° 50′	.9′
8.75	.022	6° 33′		175	175.3	0.2	3° 43′		12.75	.042	4° 29′	.2′	255	255.2	0.2	1° 45′	.9′

The ordinates between rails, straight or curved, for turn-outs to inside or outside, at ¼, ½, and ¾ of the chord-distance, k, are 4.13, 7.50, and 9.38. For any other distance, $(\mathbf{AB})=i$, multiply the chords and ordinates, and divide the degree, $D\sigma$, by one-tenth of $(\mathbf{AB})=i$.

WITCHES.

A		Fa = F, Fb = I					Switches				
		14			16		I=S Single			I = 2 S Double	
		$c-2g$ .. 5.8			....> 6.58		<..$i=(g-s)$..> 4.29			<..$i=(g-2s)$..> 3.87	
Gauge (g=4.708); Throw (s=0.417)		$\frac{}{Fb}+.2$			$\frac{753}{Fa+Fb}+.3$		$\frac{491}{F+S}+.2$			$\frac{443}{F''+_2 S}+.1$	
		.43	2.00		6.16	4.93	2.88	4.02	3.22	1.88	3.63	2.90	1.69
		$-Fb)+k$					100×(F−S)+k			100 (F″−2 S)+k	
		a^2-Fb^2) .13 (Fa^2-Fb^2)					.204 (F^2-S^2)			.226 ($F''^2-_4 S^2$)	
		$F''^2=.902\ (Fa^2+Fb^2)+2.20\ S^2$									

B		Fa = F, Fb = I					Switches				
		12			14		I = S Single			I=2 S Double	
		$c-2g$ .. 6			....> 8		<..$i=(g-s)$..> 2.67			<..$i=(g-2s)$..> 2.33	
Gauge (g=3.00); Throw (s = 0.33)		$\frac{}{Fb}+.2$			$\frac{915}{Fa+Fb}+.3$		$\frac{305}{F+S}+.2$			$\frac{217}{F+S}+.1$	
		.50	2.62		7.50	6.00	3.50	2.50	2.00	1.16	2.18	1.75	1.02
		a^2-Fb^2)			.109(Fa^2-Fb^2)			.327 (F^2-S^2)			.375 ($F''^2-_4 S^2$)		
		$F''^2=.872\ (Fa^2+Fb^2)+2.26\ S^2$											

C	S	8°			9°			10°			11°		
	Multipliers	.62	3.50	2.62	2.95	3.94	2.95	3.33	4.38	3.33	3.61	4.81	3.61

* For Crossings,

† When I is gre S for double switches, and to the lesser frog-angle b for crossings, and subtrac F, and S.

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are the same, and the resulting formulæ from table IX. will be as follows:

	Standard Gauge.	Narrow Gauge.	
$l =$	$2.8 \times n$	$2 \times n$	(7)
$k =$	$6.61 \times n$	$4 \times n$	(8)
$D\delta =$	$610 + n^2 \pm D\alpha$	$950 + n^2 \pm D\alpha$	(9)
$n'' =$	$7 \times n + 10$	$7 \times n + 10$	(10)
$k'' =$	$3.86 \times n$	$2.24 \times n$	(11)

In these formulæ, k is the distance from the toe of the switch, and n the number, of the main-track frog; $D\delta$ is the degree of curvature of the switch and turn-out rails; $D\alpha$ the degree of the main track, plus when the turn-out is to the inside, and minus when it is to the outside; n'' the number, and k'' the distance, of the middle frog for a double turn-out, in which case the numbers of the main-track frogs must be the same. The middle frog will be in the centre of the track.

As before stated, the diagram in plate XIV., described in article 2, problem XVIII., will apply in the case under consideration by giving to l and S their values from (4) and (5).

A diagram may also be constructed by making the throw, s, and the switch-angle, S, equal to zero, in the formulæ of article 2; but in such a diagram the frog-distances will be estimated from the heel of the switch-rail and not from the toe, and the length, l, of the switch-rail must be as given in (4) or (7). On such a diagram the main-track and turn-out curves might be drawn, and the frog-circles (best drawn to correspond to the frog-numbers) would give the positions of the main-track and middle frogs, and also the degrees of the turn-out curves corresponding to all curvatures of the main track; but in the case of a double turn-out the numbers of the two frogs on the main-track rails must be the same.

Figure	C	en b; to find Ia and Ib		$a \pm b$
			½ Ia	
		cot. O	± ½ I ± ½ Ib	$a \pm b = \frac{\frac{1}{2} i \sin. O}{\sin. \frac{1}{2} I b \sin \frac{1}{2} I a}$
80	C	cot. ½ (A′+B′)	+ ½ Ib − ½ (A′−B′)	$a - b = \frac{\frac{1}{2} i \sin. \frac{1}{2} (A'+B')}{\sin. \frac{1}{2} I a \sin. \frac{1}{2} I b}$
81		"	+ ½ Ib + ½ (A′−B′)	$b - a =$ "
81½		"	+ ½ Ib + ½ (A′−B′)	$a - b =$ "
82	E	"	− ½ Ib + ½ (A′−B′)	$a + b =$ "
83		"	+ ½ Ib + ½ (A′−B′)	$a + b =$ "
84		tan. ½ (A′+B′)	− ½ Ib + 90° + ½ (A′−B′)	$a + b = \frac{\frac{1}{2} i \cos. \frac{1}{2} (A'+B')}{\sin. \frac{1}{2} I a \sin. \frac{1}{2} I b}$
85	C	cot. ½ (A′−B′)	− ½ Ib + ½ (A′+B′)	$b - a = \frac{\frac{1}{2} i \sin. \frac{1}{2} (A'-B')}{\sin. \frac{1}{2} I a \sin. \frac{1}{2} I b}$
86		"	+ ½ Ib + ½ (A′+B′)	$a - b =$ "
87	F	"	+ ½ Ib + ½ (A′+B′)	$a + b =$ "
88	C	an. ½ (A′−B′)	+ ½ Ib − 90° + ½ (A′+B′)	$a - b = \frac{\frac{1}{2} i \cos. \frac{1}{2} (A'-B')}{\sin. \frac{1}{2} I a \sin. \frac{1}{2} I b}$
88¾		"	+ ½ Ib + 90° − ½ (A′+B′)	$b - a =$ "
89		"	− ½ Ib + 90° − ½ (A′+B′)	$a + b =$ "

TABLE XI. (A), PROBLEM XXIII.

REVERSED CURVES WITH EQUAL RADII.

TABLE A applies for any value of the common tangent t.

$$R=+\left(1-\frac{t^2}{i^2}\right) \quad (1) \qquad S=\pm\frac{\frac{1}{2}(\text{sin. A}\pm\text{sin. B})}{\text{sin. }\frac{1}{2}\text{ I}} \quad (2) \qquad r=\frac{\frac{1}{2}i}{\text{sin. }\frac{1}{2}\text{ I}}\cdot\left(\sqrt{+R+S^2}\pm S\right) \quad (3) \qquad \tan.\ T=\frac{\frac{1}{2}t}{r} \quad (4)$$

$$\cos.\ C=\cos.\ T\times\tfrac{1}{2}(\cos.\ B\pm\cos.\ A) \quad (5) \qquad Ia=(C\pm T)\pm A';\ Ib=(C\pm T)\pm B \quad (6)$$

Plate	Figure	I= ±Ia±Ib	S	Cos. C	Ia	Ib
19	102	(A′+B′)=Ia−Ib	$-\frac{\frac{1}{2}(\text{sin. A}'-\text{sin. B}')}{\text{sin. }\frac{1}{2}(\text{A}'+\text{B}')}$	cos. T. ½ (cos. B′+cos. A′)	(C′−T′)+A′	(C′−T′)−B′
19	101	(A′−B′)=Ia−Ib	$-\frac{\frac{1}{2}(\text{sin. A}'+\text{sin. B}')}{\text{sin. }\frac{1}{2}(\text{A}'-\text{B}')}$	"	(C′−T′)+A′	(C′−T′)+B′
20	106	180°+(A′+B′)=Ia+Ib	$-\frac{\frac{1}{2}(\text{sin. A}'+\text{sin. B}')}{\cos.\ \frac{1}{2}(\text{A}'+\text{B}')}$	cos. T. ½ (cos. B′−cos. A′)	180°−(C′±T′)+A′	(C′±T′)+B′
20	107	180°−(A′+B′)=Ia+Ib	+ "	"	180°−(C′±T′)−A′	(C′±T′)−B′
20	108	180°−(A′−B′)=Ia+Ib	$+\frac{\frac{1}{2}(\text{sin. A}'-\text{sin. B}')}{\cos.\ \frac{1}{2}(\text{A}'-\text{B}')}$	"	180°−(C′±T′)−A′	(C′±T′)+B′
21	109	180°+(A′−B′)=Ia+Ib	"	"	180°+(C′±T′)+A′	(C′±T′)−B′

TABLE XI. (B), PROBLEM XXIII.

REVERSED CURVES WITH EQUAL RADII.

TABLE B applies when the common tangent t is equal to zero.

$$\cos. C = \tfrac{1}{2}(\cos. B' \pm \cos. A'). \quad (1) \qquad I\alpha = C \pm A;\ I\beta = B \pm C. \quad (2) \qquad r = \frac{\frac{1}{4} i \sin. \frac{1}{2}(I \mp B)}{\sin. \frac{1}{2} I\alpha \sin. \frac{1}{2} I\beta} \quad (3)$$

Plate	Figure	$I = \pm I\alpha \pm I\beta$	Cos. C′	$I\alpha$	$I\beta$	r
17	87	$(A' + B') = I\alpha - I\beta$	$\frac{1}{2}(\cos. B' + \cos. A')$	$C' + A'$	$C' - B'$	$\frac{\frac{1}{4} i \sin. \frac{1}{2}(A' - B')}{\sin. \frac{1}{2} I\alpha \sin. \frac{1}{2} I\beta}$
17	83	$(A' - B') = I\alpha - I\beta$	"	$C' + A'$	$C' + B'$	$\frac{\frac{1}{4} i \sin. \frac{1}{2}(A' + B')}{\sin. \frac{1}{2} I\alpha \sin. \frac{1}{2} I\beta}$
17	89¼	$180° + (A' + B') = I\alpha + I\beta$	$\frac{1}{2}(\cos. B' - \cos. A')$	$180° - C' + A'$	$C' + B'$	$\frac{\frac{1}{4} i \cos. \frac{1}{2}(A' - B')}{\sin. \frac{1}{2} I\alpha \sin. \frac{1}{2} I\beta}$
17	89	$180° - (A' + B') = I\alpha + I\beta$	"	$180° - C' - A'$	$C' - B'$	"
20	108	$180° - (A' - B') = I\alpha + I\beta$	"	$180° - C' - A'$	$C' + B'$	$\frac{\frac{1}{4} i \cos. \frac{1}{2}(A' + B')}{\sin. \frac{1}{2} I\alpha \sin. \frac{1}{2} I\beta}$
17	84	$180° + (A' - B') = I\beta + I\alpha$	"	$180° - C' + A'$	$C' - B'$	"

ON TANGENT.

$$\text{Tan. T} = \frac{t}{(a \pm b)}. \quad (3)$$

$$t = (a \pm b) \text{ tan. T} = \frac{\Sigma d \text{ sin. T}}{\text{sin. C}} = \frac{\Sigma l \text{ sin. T}}{\text{cos. C}'}. \quad (4) \quad (5)$$

Plate	Figure		Σd	Ia	Ib
19	102	(A	$-a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+(C'-T')+A'$	$-B'+(C'-T')$
19	103	(A	$+a \text{ sin } A' \overset{+}{-} b \text{ sin. } B'$	$-(C' \pm T')+A'$	$+B'+(C' \pm T')$
20	104	(A	$-a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+(C'-T')+A'$	$-B'-(C'-T')$
20	105	(A	$+a \text{ sin. } A' \overset{+}{-} b \text{ sin } B'$	$+(C \pm T')-A'$	$-B+(C' \pm T')$
20	106	180°+(A	$-a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+180°-(C' \pm T')+A'$	$+B'+(C' \pm T')$
20	107	180°−(A	$+a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+180°-(C' \pm T')-A'$	$-B'+(C' \pm T')$
20	108	180°−(A	$+a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+180°-(C' \pm T')-A'$	$+B'-(C' \pm T')$
21	109	180°+(A	$+a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+180°-(C' \pm T')+A'$	$-B'+(C' \pm T')$
21	110	(A	$-a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$+A'-(C' \pm T')$	$B'+(C' \pm T')$
17	85	(A	$-a \text{ sin. } A' \overset{-}{-} b \text{ sin. } B'$	$A'+C'$	$B'-C'$
17	86	(A	$-a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$A'+C'$	$B'-C'$
22	114	(A	$+a \text{ sin. } A' \overset{+}{-} b \text{ sin. } B'$	$(C' \pm T')-A'$	$-B'-(C' \pm T')$
16	80	(A'	$-a \text{ sin. } A' \overset{-}{+} b \text{ sin. } B'$	$C'-A'$	$C'-B'$
16	81	(A	$+a \text{ sin. } A' \overset{-}{-} b \text{ sin. } B'$	$C'+A'$	$C'+B'$
16	81½	(A	$-a \text{ sin. } A' \overset{+}{+} b \text{ sin. } B'$	$C'+A'$	$C'+B'$

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THE ATTENTION

OF

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IS CALLED TO

NIXON'S PATENT

Course-Reading Attachment

TO

SURVEYORS' AND ENGINEERS'

TRANSITS.

PATENTED FEBRUARY 19, 1884.

This instrument does away *entirely* with the calculation of courses, as it indicates the course of every line run by it and reads the same down to minutes.

It is very simple, and yet the most important improvement ever made in the manufacture of the Transit. This attachment can be put on any manufacturer's instrument, and adds but little to the cost, and yet will save time enough every day to pay for itself. Arrangements made with instrument-makers to manufacture on a royalty; also instruments for sale.

Apply to the patentee,

T. L. NIXON,

Lock Box 75. **NEW TACOMA,**

Washington Territory.